Advanced Finite Element Methods

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Contents

Chapter 1

The classical finite element method (FEM)

1.1 Principle of the method

For solving a problem on a computer that can only store a finite amount of information a discrete form of the problem is needed. In the Finite Difference method one simply computes an approximation of the solution at a finite number of grid points. In the Finite Element method, which is mathematically more involved, the idea is to look for the solution in a finite dimensional vector space, *i.e.* for some well chosen vector space V_h, with basis $(\varphi_i)_{0\leq i\leq N-1}$, the approximate solution has the form

$$u_h(x) = \sum_{i=0}^{N-1} u_i \varphi_i(x).$$

The basis being given, the approximate solution u_h is fully determined by its coefficients u_i in this basis, which need not be values of u_h at some points in the computational domain, but can be in some cases.

The question now becomes how to choose V_h and determine the coefficients u_i such that u_h is a good approximation of the solution u of the original problem. Let us consider as a first example the Poisson problem with homogeneous Dirichlet boundary conditions:

$$-\Delta u = f, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega. \tag{1.1}$$

The first idea, introduced by Ritz in his thesis in Göttingen in 1902, was to transform the boundary problem into an equivalent minimisation problem. Indeed, via the Dirichlet principle (1.1) is equivalent to the minimisation problem

$$\min_{u\in H_0^1(\Omega)} \left(\frac{1}{2} \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x \right). \tag{1.2}$$

We shall need the following Hilbert spaces, defined for a domain $\Omega \in \mathbb{R}^d$

$$H^1(\Omega) = \{u \in L^2(\Omega),\ \nabla u \in (L^2(\Omega))^d\}, \qquad H_0^1(\Omega) = \{u \in H^1(\Omega),\ u = 0 \text{ on } \partial\Omega\}.$$

The scalar product associated to these Hilbert spaces is

$$(u,v)_1 = \int_\Omega \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_\Omega u(x)v(x)\,\mathrm{d}x.$$

Moreover, we denote by

$$\|u\|_{1,\Omega} = \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x + \int_\Omega u(x)^2\,\mathrm{d}x,\ \ |u|_{1,\Omega} = \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x,\ \ \|u\|_{0,\Omega} = \int_\Omega u(x)^2\,\mathrm{d}x,$$

the norm in $H^1(\Omega)$, the semi-norm in $H^1(\Omega)$ and the norm in $L^2(\Omega)$. Then, the original problem being transformed into a minimisation problem, it becomes quite natural to look for an approximation in a finite dimensional subspace of the function space in which the minimisation problem is posed (in our case $H_0^1(\Omega)$), which means that the minimisation is performed by considering only minima in a finite dimensional subspace. Then if the form of finite dimensional space is chosen such that any function of the original space can be approximated to any given tolerance, by a function of the approximation space, we should be able to get a good approximation. Ritz who was actually looking at solutions for the bilaplacian equation, chose as basis functions for V_h a finite number of eigenfunctions of his operator.

The standard method to solve a minimisation problem with a cost functional J defined on a Hilbert space V, of the form

$$\min_{u\in V} J[u],$$

is to solve the associated Euler equation $J'[u] = 0$ obtained by computing the Fréchet derivative of the functional that we want to minimise. Note that the Fréchet derivative gives a rigorous definition of the functional derivative used in physics for functions that are in a Banach (including Hilbert) space. Consider a functional J from a Hilbert space V into $\mathbb{R}$. Its Fréchet derivative J', assuming it exists, is a linear form on V, which means that it maps any function from V to a scalar. It can be computed using the Gâteaux formula:

$$J'[u](v) = \lim_{\varepsilon\to 0} \frac{J[u+\epsilon v] - J[u]}{\varepsilon}. \tag{1.3}$$

Let us apply this formula to our problem for which

$$J[u] = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x.$$

We have for any $v \in V = H_0^1(\Omega)$

$$\begin{aligned} J[u+\varepsilon v] &= \frac{1}{2}\int_\Omega |\nabla u(x) + \varepsilon v(x)|^2\,\mathrm{d}x - \int_\Omega f(x)(u(x)+\varepsilon v(x))\,\mathrm{d}x \\ &= \frac{1}{2}\left(\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x + 2\varepsilon\int_\Omega \nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x + \varepsilon^2\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x\right) \\ &\quad - \int_\Omega f(x)u(x)\,\mathrm{d}x - \varepsilon\int_\Omega f(x)v(x)\,\mathrm{d}x. \end{aligned}$$

From which we deduce, using the Gâteaux formula (1.3) that

$$J'[u](v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) \, \mathrm{d}x - \int_\Omega f(x) v(x) \, \mathrm{d}x.$$

Note that $J'[u]$ being a linear form on V is defined by applying it to some vector $v \in V$. Finally the solution of our minimisation problem (1.2), is a solution of the Euler equation $J'[u] = 0$ or equivalently $J'[u](v) = 0$ for all $v \in V$, which reads

$$\int_\Omega \nabla u(x) \cdot \nabla v(x) \, \mathrm{d}x = \int_\Omega f(x) v(x) \, \mathrm{d}x \quad \forall v \in H_0^1(\Omega). \tag{1.4}$$

This is also what is called the variational formulation, or the weak formulation of the original boundary value problem (1.1). Note that this variational formulation expresses in some sense the orthogonality of the residual to the space in which the solution is sought. This is more general than Euler equations of minimisation problems as noticed by Galerkin and has a wide range of applications. One can even extend this concept by making the residual orthogonal to a different function space, than the one in which the solution lives. Such methods are called Petrov-Galerkin methods and are beyond the scope of this lecture.

So the principle of the Galerkin (or Ritz-Galerkin) method is to look for a solution in a finite dimensional subspace $V_h \subset V$ of the original space and to use the same variational formulation (1.4) as the one defining the exact solution, with test functions also in V_h to characterise the solution. What remains to be done now is to choose V_h with good approximation properties. As we will see later, the stability of the Galerkin method follows directly from the well-posedness of the variational problem (1.4).

The finite dimensional space V_h is in general defined by its basis functions. For those, Ritz used eigenfunctions of the problem. But those are in general cumbersome to compute. Galerkin proposed to use general classes of simple functions, trigonometric functions or polynomials, that are know to be able to approximate any continuous function with a finite number of basis functions. Trigonometric polynomials which are linked to Fourier series are very good in periodic domains, with a few simple extensions. Polynomials enjoy more widespread applications, however to get a good conditioning of the linear system that is obtained at the end, care needs to be taken in the choice of the basis functions. The monomial basis $(1, x, x^2, \dots)$ has very bad properties. Best approximations are provided by the orthogonal Legendre polynomials or by the Chebyshev polynomials which are used in practice. Note that all the basis functions we have mentioned up to now have a global support in the computational domain and thus lead to full matrices in the linear system, which can be computationally expensive. Methods using such bases are actually not known as Finite Element methods but rather as spectral methods.

Another ingredient is needed to define what is known as Finite Element methods. This was introduced by Courant in 1943 [7] and consists in using basis functions with a small support in the computational domain, so that its product with other basis functions vanishes for most of the other basis functions leading

to a very sparse matrix, *i.e.* with very few non zero terms, in the linear system, which can be solved very efficiently on a computer. To this aim, the computational domain is decomposed into small elements, in general triangles or quads in 2D and the basis functions are chosen to be relatively low order polynomials, on each of these elements. Convergence being achieved by taking smaller elements like the cells in the Finite Difference method. In 1D a finite element mesh will look like a finite difference mesh. An example of an unstructured Finite Element mesh in 2D is displayed in Figure 1.1, which shows the great flexibility in particular to handle complicated boundaries with finite elements, which finite differences do not provide. This is a key to its very wide usage.

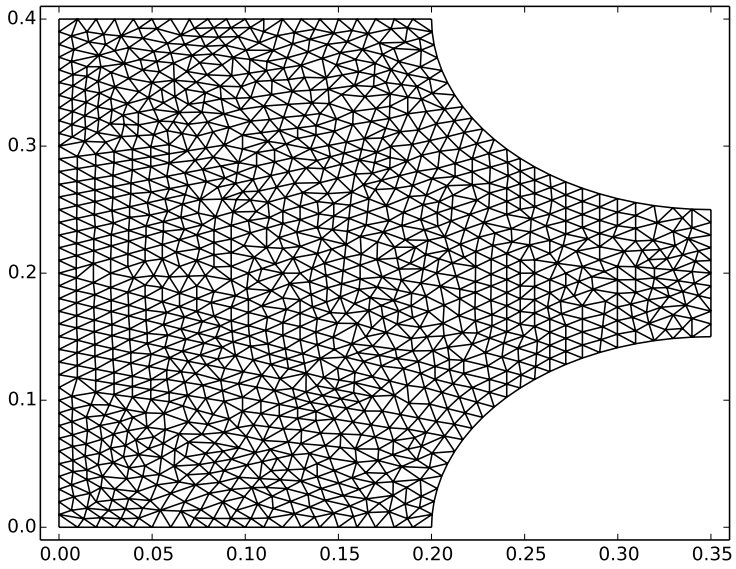

Figure 1.1: Example of a 2D finite element mesh consisting of triangles.

The article by Gander and Wanner [12] provides a clear and well documented overview of the historical developments of the Finite Element method. For more technical historical developments of the Finite Difference and Finite Element methods on can also consult [15].

In summary, the finite element method consists in looking for a solution of a variational problem like (1.4), in a finite dimensional subspace V_h of the space V where the exact solution is defined. The space V_h is characterised by a basis $(\varphi_1, \ldots, \varphi_N)$ so that finding the solution of the variational problem amounts to solving a linear system. Indeed, express the trial function u_h and the test function $\tilde{u}_h$ on this basis:

$$u_h(x) = \sum_{j=1}^{N} u_j \varphi_j(x), \qquad \tilde{u}_h(x) = \sum_{i=1}^{N} \tilde{u}_i \varphi_i(x),$$

and plug these expressions in the variational problem (1.4). This yields

$$\sum_{i=1}^{N}\sum_{j=1}^{N} u_j\tilde{u}_i \int_\Omega \nabla\varphi_i(x)\cdot\nabla\varphi_j(x)\,\mathrm{d}x = \sum_{i=1}^{N}\tilde{u}_i\int_\Omega f(x)\varphi_i(x)\,\mathrm{d}x.$$

This can be expressed in matrix form, $\tilde{U}_h^\top A_h U_h = \tilde{U}_h^\top b_h$, which is equivalent to the linear system $A_h U_h = b_h$ as the previous equality is true for all $\tilde{U}_h$, where

$$U_h = (u_1,\ldots,u_N)^\top,\ \tilde{U}_h = (\tilde{u}_1,\ldots,\tilde{u}_N)^\top,\ b_h = (\int_\Omega f(x)\varphi_1(x)\,\mathrm{d}x,\ldots,\int_\Omega f(x)\varphi_N(x)\,\mathrm{d}x)^\top$$

and the matrix A_h is defined by its entries which are

$$(\int_\Omega \nabla\varphi_i(x)\cdot\nabla\varphi_j(x)\,\mathrm{d}x)_{1\le i,j\le N}.$$

In order to completely determine the Finite Element problem, we still need to construct a Finite Element space V_h of dimension N and a basis for it ($\varphi_1,\cdot$, $varphi_N$). In some case, for simple domains, one can use a global basis, like a truncated Fourier expansion, or Chebyshev polynomials. However these have the drawback of having a support on the full domain leading to dense approximation matrices, and also of being limited to simple typically rectangular domains. Therefore one generally constructs a basis by first defining a mesh of the computational domain and then defining the basis function locally on each cell, using continuity requirements at the cell interfaces to enforce $V_h \subset V$, where V is the function space where the original variational problem is posed.

A simple example is the case of locally Lagrange polynomials in 1D. Consider the interval $[a,b]$ and subdivide it into a mesh $a = x_0 < x_1 < \cdots < x_{M-1} < x_M = b$. The mesh can be uniform or not. The function space we are approximating is $H_0^1([a,b])$. Define now the interpolation points, consisting of all interior grid points and midpoints of each cell:

$$y_1 = \frac{x_0+x_1}{2}, y_2 = x_1, y_3 = \frac{x_1+x_2}{2}, y_4 = x_2, \ldots y_{N-1} = x_{M-1}, y_N = \frac{x_{M-1}+x_M}{2},$$

where $N = 2M-1$. We can know define the basis functions locally on each cell as being the piecewise polynomial such that $\varphi_i(y_j) = \delta_{ij}$, where the Kronecker symbol δ_{ij} is one if $i=j$ and zero else. This correspond in each cell to a Lagrange basis polynomial. Each basis function is thus associated to one interpolation point, which is called *degree of freedom* in the Finite Element terminology. Note that the basis functions associated to a cell mid-point have a support restricted to the corresponding cell, *i.e.* they are zero in all the other cells. On the other hand, the basis functions associated to a grid point have a support of the two cell sharing this grid points. Because $V_h \subset H_0^1$ all the basis functions of V_h need to vanish at the endpoints a and b of the computational domain. Therefore there is no degree of freedom at the end points and there are correspondingly no basis functions associated to the end points.

1.2 Basic properties of the Sobolev spaces $H^m(\Omega)$, $H(\mathbf{div},\Omega)$ and $H(\mathbf{div},\Omega)$

In the course of the lecture we shall work with the Sobolev spaces $H^m(\Omega)$, $H(\mathrm{div},\Omega)$ and $H(\mathrm{div},\Omega)$ and recall here there basic properties without proof. For a more detailed presentation with proofs we refer to Section 2.1. of [5].

1.2.1 The Sobolev space H^m

For any integer $m \geq 1$, one can define

$$H^m(\Omega) = \{v \in L^2(\Omega) \mid D^\alpha v \in L^2(\Omega),\ |\alpha| \leq m\} \tag{1.5}$$

The most classical second order operator is the Laplace operator, which reads in an arbitrary dimension d (generally $d = 1, 2$ or 3),

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_d^2}.$$

The classical Green formula for the Laplace operator reads

For essential boundary conditions related with this Green formula we shall define the space

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v_{|\partial\Omega} = 0\}.$$

And other classical operator which comes from elasticity is the bilaplacian operator $\Delta^2 = \Delta\Delta$, which is a fourth order operator. The Green formula needed for variational formulations of PDEs based on the bilaplacian reads

$$\int_\Omega \Delta^2 u\, v \,\mathrm{d}\mathbf{x} = \int_\Omega u\, \Delta^2 v \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \left(u\frac{\partial \Delta v}{\partial n} - v\frac{\partial \Delta u}{\partial n} + \Delta u \frac{\partial v}{\partial n} - \Delta v \frac{\partial u}{\partial n} \right) \mathrm{d}\sigma \tag{1.6}$$

$$H_0^2(\Omega) = \{v \in H^1(\Omega) \mid v_{|\partial\Omega} = 0,\ \frac{\partial v}{\partial n}\big|_{\partial\Omega} = 0\}.$$

1.2.2 The Sobolev space $H(\mathbf{div},\Omega)$

$$\int_\Omega \nabla \cdot \mathbf{u}\, v \,\mathrm{d}x = -\int_\Omega \mathbf{u} \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \mathbf{u} \cdot \boldsymbol{\nu}\, v \,\mathrm{d}\sigma \tag{1.7}$$

1.2.3 The Sobolev space $H(\mathbf{curl},\Omega)$

The Green's formula that will be useful for variational problems involving the curl operator reads

$$\int_\Omega \mathbf{u} \cdot \nabla \times \mathbf{v} = \int_\Omega \nabla \times \mathbf{u} \cdot \mathbf{v} + \int_{\partial\Omega} (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{v} \,\mathrm{d}s. \tag{1.8}$$

1.3 The variational (or weak) form of a boundary value problem

The variational form of a boundary value problem contains all its elements, which are the partial differential equation in the interior of the domain and the boundary conditions. There are two very distinct ways to handle the boundary conditions depending on how they appear when deriving the variational formulation. If they appear on the test function they are called *essential boundary conditions* and need to be included in the space where the solution is looked for. If they appear on the trial function, which will be the approximate solution, they can be handled in a natural way in the variational formulation. Such boundary conditions are called *natural boundary conditions*. We will see on the examples of Dirichlet and Neumann boundary conditions how this works in practice.

In order to define the variational formulation, we will need the following Green formula: For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$

$$-\int_\Omega \Delta u\, v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \,\mathrm{d}\sigma. \tag{1.9}$$

Here $H^2(\Omega)$ denotes the Hilbert space of the functions whose partial derivatives up to second order are in $L^2(\Omega)$ and $\frac{\partial u}{\partial n} = \nabla u \cdot \boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is the outbound normal at any point of the boundary.

Remark 1 *Functions in Sobolev spaces are not necessarily continuous and hence their value on the boundary is not naturally defined. In fact, one can define, what is called a* trace *on the boundary, by first defining it for sufficiently smooth functions and then extending it by continuity, typically using a Green's formula like* (1.9)*. So the Green's formula also tells us what kind of trace can be defined. For a more detailed description one can consult a classical functional analysis or a Finite Element textbook like for example [5].*

1.3.1 Case of homogeneous Dirichlet boundary conditions

Let $f \in L^2(\Omega)$. Consider the boundary value problem

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.10}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.11}$$

Assume that $u \in H^2(\Omega)$, multiply (1.10) by $v \in H^1(\Omega)$ and integrate using the Green formula (1.9), which yields

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \,\mathrm{d}\sigma = \int_\Omega f v \,\mathrm{d}x.$$

Here u does not appear in the boundary integral, so we cannot apply the boundary condition directly. But in the end u will be in the same function space as the test

function v, which appears directly in the boundary integral. This is the case of an essential boundary condition. So we take test functions v vanishing on the boundary. We then get the following variational formulation:
Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \quad \forall v \in H_0^1(\Omega). \tag{1.12}$$

The solutions of this variational formulation are called *weak solutions* of the original boundary value problem. The solutions which are also in $H^2(\Omega)$ are called *strong solutions*. Indeed we can prove that such a solution is also a solution of the initial boundary value problem (1.10)-(1.11). If $u \in H^2(\Omega)$, the Green formula (1.9) can be used, and as φ vanishes on the boundary it yields

$$-\int_\Omega \Delta u \, \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \quad \forall v \in H_0^1(\Omega).$$

This implies, as $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, that $-\Delta u = f$ in $L^2(\Omega)$ and so almost everywhere. On the other hand as $u \in H_0^1(\Omega)$, $u = 0$ on $\partial\Omega$. So u is a strong solution of (1.10)-(1.11).

1.3.2 Case of non homogeneous Dirichlet boundary conditions

Let $f \in L^2(\Omega)$ and $u_0 \in H^1(\Omega)$. We consider the problem

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.13}$$

$$u = u_0 \quad \text{on } \partial\Omega. \tag{1.14}$$

As the value of u on the boundary cannot be directly put in the function space if it is not zero, as else the function space would not be stable by linear combinations, we need to bring the problem back to the homogeneous case. To this aim let $\tilde{u} = u - u_0$. We then show as previously that $\tilde{u}$ is a solution of the variational problem
Find $\tilde{u} \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla \tilde{u} \cdot \nabla v \, dx = \int_\Omega f v - \int_\Omega \nabla u_0 \cdot \nabla v \, \mathrm{d}x \quad \forall v \in H_0^1(\Omega). \tag{1.15}$$

This is the variational problem that needs to be solved for non homogeneous Dirichlet boundary conditions. In practice u_0 is only known on the boundary and needs to be extended to the whole domain. For standard finite elements one can pick the degrees of freedom on the boundary to be the projection (or finite element interpolation) of u_0 on the boundary and let the inner degrees of freedom be zero.

1.3.3 Case of Neumann boundary conditions

Let $f \in L^2(\Omega)$ and $g \in H^1(\Omega)$. We consider the problem

$$-\Delta u + u = f \quad \text{in } \Omega, \tag{1.16}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega. \tag{1.17}$$

Assuming that $u \in H^2(\Omega)$, we multiply by a test function $v \in H^1(\Omega)$ and integrate using the Green formula (1.9), which yields

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, \mathrm{d}\sigma + \int_\Omega uv \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x.$$

Replacing $\frac{\partial u}{\partial n}$ by its value g on the boundary, we obtain the variational formulation *Find* $u \in H^1(\Omega)$ *such that*

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} + \int_\Omega uv \, \mathrm{d}\mathbf{x} = \int_\Omega fv \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} gv \, \mathrm{d}\sigma \quad \forall v \in H^1(\Omega). \tag{1.18}$$

Let us now show that u is a strong solution of the boundary value problem, provided it is in $H^2(\Omega)$. As $H_0^1(\Omega) \subset H^1(\Omega)$ one can first take only test functions in $H_0^1(\Omega)$. Then as in the case of homogeneous Dirichlet conditions it follows from the Green formula (1.9) that

$$\int_\Omega (-\Delta u + u)\, \varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x \qquad \forall \varphi \in H_0^1(\Omega).$$

This implies, as $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, that $-\Delta u + u = f$ in $L^2(\Omega)$ and so almost everywhere.

It now remains to verify that we have the boundary condition

$$\frac{\partial u}{\partial n} = g \text{ on } \partial\Omega.$$

For that we start from (1.18) and apply the Green formula (1.9), which yields

$$-\int_\Omega \Delta u \, v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} \, v \, \mathrm{d}\sigma + \int_\Omega uv \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x + \int_{\partial\Omega} gv \, \mathrm{d}\sigma \quad \forall v \in H^1(\Omega),$$

and as $-\Delta u + u = f$, it remains

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \, v \, \mathrm{d}\sigma = \int_{\partial\Omega} gv \, \mathrm{d}\sigma \quad \forall v \in H^1(\Omega),$$

which yields that $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$.

1.4 Solution of the the variational formulation

The variational problems we consider in this chapter, as the examples from the previous section, can be written in the following abstract form
Find $u \in V$ such that

$$a(u,v) = l(v) \quad \forall v \in V, \tag{1.19}$$

where V is a Hilbert space, a is a symmetric continuous and coercive bilinear form and l a continuous linear form.

The most convenient tool for provient existence and uniqueness of the solution of a variational problem is the Lax-Milgram theorem that we recall here:

Theorem 1 (Lax-Milgram) *Let V a Hilbert space with the norm $\|.\|_V$. Let $a(.,.)$ be a continuous, symmetric and coercive bilinear form on $V \times V$, i.e.*

1. *(Continuity): there exists C such that for all $u, v \in V$*

$$|a(u,v)| \leq C\|u\|_V\|v\|_V.$$

2. *(Coercivity): there exists a constant $\alpha > 0$ such that for all $u \in V$*

$$a(u,u) > \alpha\|u\|_V^2.$$

Let $l(.)$ a continuous linear form on V, i.e. there exists C such that for all $v \in V$

$$|l(v)| \leq C\|v\|_V.$$

Then there exists a unique $u \in V$ such that

$$a(u,v) = l(v) \quad \forall v \in V.$$

To illustrate the use of the Lax-Milgram theorem, let us apply it to the examples of the previous section. The case of the problem with homogeneous Dirichlet boundary conditions, yields to the variational formulation (1.12), which fits into the framework of the Lax-Milgram theorem with $V = H_0^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$, $l(v) = \int_\Omega fv \, \mathrm{d}x$. The continuity of the bilinear form a and the linear form l follows from the Cauchy-Schwarz inequality. a is clearly symmetric and the coercivity is a consequence of the Poincaré inequality:

$$\|u\|_{0,\Omega} \leq C|u|_{1,\Omega} \quad \forall u \in H_0^1(\Omega), \tag{1.20}$$

which implies that

$$a(u,u) = |u|_{1,\Omega} \geq \frac{1}{2}|u|_{1,\Omega} + \frac{1}{2C}|u|_{0,\Omega} \geq \min(\frac{1}{2}, \frac{1}{2C})\|u\|_{1,\Omega}.$$

The case of Neumann boundary conditions yields the variational formulation (1.18), in which case $V = H^1(\Omega)$, a is the H_1 scalar product which is clearly continuous, symmetric and coercive, and $l(v) = \int_\Omega fv \, \mathrm{d}x + \int_{\partial\Omega} gv \, \mathrm{d}\sigma$, which is a continuous linear form thanks to the Cauchy-Schwarz inequality and the continuity of the trace.

1.5 Description of a Finite Element

1.5.1 Formal definition of a Finite Element

Let (K, P, Σ) be a triple such that
(i) K is a closed subset of $\mathbb{R}^n$ of non empty interior,
(ii) P is a finite dimensional vector space of functions defined on K,
(iii) Σ is a set of linear forms on P of finite cardinal N, $\Sigma = \{\sigma_1, \ldots, \sigma_N\}$.

Definition 1 *The elements of* Σ *are called* degrees of freedom *of the Finite Element.*

The degrees of freedom characterise the basis of P associated to the Finite Element.

Definition 2 Σ *is said to be* P***-unisolvent*** *if for any* N*-tuple* $(\alpha_1, \ldots, \alpha_N)$*, there exists a unique element* $p \in P$ *such that* $\sigma_i(p) = \alpha_i$ *pour* $i = 1, \ldots, N$*.*

Definition 3 *The triple* (K, P, Σ) *of* $\mathbb{R}^n$ *is called* ***Finite Element*** *of* $\mathbb{R}^n$ *if it satisfies (i), (ii) and (iii) and if* Σ *is* P*-unisolvent.*

In the definition of a Finite Element, K is the domain on which the Finite Element is defined, P is the finite dimensional approximation space, and Σ uniquely defines a basis of P, which is needed to build the Finite Element matrices associated to the variational formulation applied to functions in the Finite Dimensional approximation space. This basis can be either defined through the degrees of freedom or directly by exhibiting an explicit formula for the basis functions. In this last case the degrees of freedom are not explicitly needed.

The unisolvence property is needed to establish that the elements of P characterised by the degrees of freedom actually form a basis of P. In practice, this is done by first verifying that the dimension is right, *i.e.* that the number of degrees of freedom is equal to $\dim P = N$ and then by checking either the injectivity or the surjectivity, which both imply the bijectivity if the dimension is right, of the mapping

$$\begin{aligned} P &\to \mathbb{R}^N \\ p &\mapsto (\sigma_1(p), \ldots, \sigma_N(p)) \end{aligned}$$

This can be formalised in the two following lemmas:

Lemma 1 *The set* Σ *is* P*-unisolvent if and only if the two following properties are satisfied*

(i) $\dim P = |\Sigma|$ *(where* $|\Sigma|$ *denotes the number of elements in the set* Σ*).*

(ii) $\sigma_j(p) = 0$ *for* $j = 1, \ldots, N \Rightarrow p = 0$*.*

Lemma 2 *The set* Σ *is* P*-unisolvent if and only if the two following properties are satisfied*

(i) $\dim P = |\Sigma| = N$.

(ii) *There exist* N *linearly independent functions* $p_i \in P$, $i = 1, \ldots, N$ *such that* $\sigma_j(p_i) = \delta_{ij}$.

In addition to its local definition, independent of its neighbours, the degrees of freedom of a Finite Element need to be chosen to allow a natural embedding into the function spaces in which the variational formulation is defined. In practice for Finite Element spaces embedded in L^2 no continuity is required, for Finite Element spaces embedded in H^1 C^0 continuity is required, for Finite Element spaces embedded in H^2 C^1 continuity is required, for Finite Element spaces embedded in $H(\mathrm{curl}, \Omega)$ C^0 continuity of the tangential component is required and for Finite Element spaces embedded in $H(\mathrm{div}, \Omega)$ C^0 continuity of the normal component is required.

We will thus present examples of Finite Elements according to their conformity, *i.e.* to the space in which they can be naturally embedded. Note that the continuity requirement is enforced by sharing the degrees of freedom on the interface between two elements and by making sure that these define uniquely the trace of P on the interface.

Remark 2 *Note that not all degrees of freedom on the interface need to be shared between the elements sharing the interface. This needs to be decided when constructing the global Finite Element space from the reference element.*

1.5.2 C^0 Lagrange Finite Elements

These are continuous Finite Elements, and the continuity will be enforced by sharing the degrees of freedom on the interface. Thanks to this continuity property the Finite Element constructed from those will be included in H^1. These elements are called H^1 *conforming*.

1D Lagrange $\mathbb{P}_k$ Element Let $a, b \in \mathbb{R}$, $a < b$. Let $K = [a, b]$, $P = \mathbb{P}_k$ the set of polynomials of degree k on $[a, b]$, $\Sigma = \{\sigma_0, \ldots, \sigma_k\}$, where $a = x_0 < x_1 < \cdots < x_k = b$ are distinct points and

$$\begin{array}{rcl} \sigma_k : P & \to & \mathbb{R}, \\ p & \mapsto & p(x_i). \end{array}$$

Moreover Σ is P-unisolvant, using Lemma 2. Indeed the Lagrange polynomials at the interpolation points $x_0, x_1, \ldots, x_k$ which read

$$l_{k,i}(x) = \frac{\displaystyle\prod_{\substack{0 \le j \le k \\ j \neq i}} (x - x_j)}{\displaystyle\prod_{\substack{0 \le j \le k \\ j \neq i}} (x_i - x_j)} \tag{1.21}$$

satisfy $l_{k,i}(x_j) = \delta_{ij}$, $0 \leq i, j \leq k$.

The fact that the points are all distinct makes the Lagrange interpolation problem well posed. The first and last point being on the boundary, will correspond to shared degrees of freedom. The corresponding basis function will be shared with the neighbouring element.

For low degree polynomials, typically up to three, the degrees of freedom can be chosen uniformly spaced. For higher degree, the resulting matrices have better properties in particular a better condition number if Gauss-Lobatto points are chosen. Note that high order Lagrange Finite Elements based on Gauss-Lobatto points are often called *spectral elements*.

The extension from one to multiple dimensions is generally done in two ways. Either one builds finite elements based on simplices, or on tensor products of 1D elements. There are other possibilities but they are less classical.

Simplex based $\mathbb{P}_k$ Finite Elements. A simplex in n dimensions is the convex hull of $n+1$ affine independent points. In 2D this is a non degenerate triangle in 3D a tetrahedron. In simplices basis functions are most easily expressed using the barycentric coordinates associated to the vertices of the simplex, that we shall denote by $\mathbf{a}_0, \ldots, \mathbf{a}_n \in \mathbb{R}^n$. The barycentric coordinate, classically denoted by $(\lambda_i)_{0 \leq i \leq n}$, associated to the vertex $\mathbf{a}_i$ is the affine function with value 1 at $\mathbf{a}_i$ and 0 at the n other vertices. More precisely, for any point $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$,

$$\lambda_i(\mathbf{x}) = \alpha_{i,0} + \sum_{l=1}^{n} \alpha_{i,l} x_l, \quad i = 0, \ldots, n. \tag{1.22}$$

The coefficients $(\alpha_{i,l})_{0 \leq l \leq n}$ being determined by the relations $\lambda_i(\mathbf{a}_j) = \delta_{ij}$.

Example: $\mathbb{P}_k$ in 2D. For 3 non aligned points $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ in $\mathbb{R}^2$ let K be the triangle defined by a_1, a_2 and a_3. Let $P = \mathbb{P}_k$ be the vector space of polynomials of degree k in 2D

$$\mathbb{P}_k = \{\mathrm{Span}(x^\alpha y^\beta), \quad (\alpha, \beta) \in \mathbb{N}^2, \quad 0 \leq \alpha + \beta \leq k\}.$$

The dimension of $\mathbb{P}_k$ is $\frac{(k+1)(k+2)}{2}$. Putting the Lagrange interpolation points on a lattice with $k+1$ points on the edges and then one point less at each level starting from one edge, we get $(k+1)+k+(k-1)+\cdots+1 = (k+1)(k+2)/2$ interpolation points and so also degrees of freedom, which is the same as the dimension of the space (see Figure 1.2 for an example with $k = 3$). The basis functions can be expressed using the barycentric coordinates. In our example with $k = 3$. The three basis functions associated to the three vertices are $(9/2)\lambda_i(\lambda_i - 1/3)(\lambda_i - 2/3)$, the basis function associated associated to the point closest to a_i on the edge $[a_i, a_j]$ is $(27/2)\lambda_i\lambda_j(\lambda_i - 1/3)$. There are six basis functions of this form. Finally the basis function associated to the center point is $27\lambda_1\lambda_2\lambda_3$. We thus find 10 basis functions for $\mathbb{P}_3$, whose dimension is 10.

Hypercube based $\mathbb{Q}_k$ Finite Element. Let K be the hypercube of dimension n $K = \prod_{1 \leq i \leq n}[a_i, b_i]$. Let $P = \mathbb{Q}_k$ be the vector space of polynomials of degree k

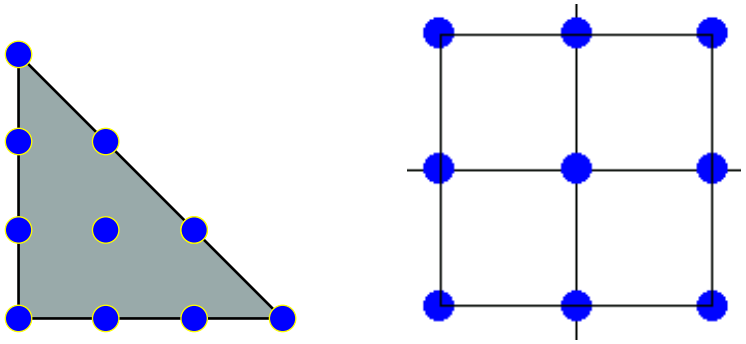

Figure 1.2: (Left) $\mathbb{P}_3$ Finite Element, (Right) $\mathbb{Q}_2$ Finite Element

in each direction:

$$\mathbb{Q}_k = \{\text{Span}(x_1^{\alpha_1} \dots x_n^{\alpha_n}), \quad (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \quad 0 \leq \alpha_1, \dots, \alpha_n \leq k\}.$$

The degrees of freedom for this element is obtained by choosing $k+1$ distinct points $a_i = x_{i,0} < \cdots < x_{i,k} = b_i$ in each direction. The corresponding basis functions are the products in all directions of all the 1D Lagrange basis functions. The Lagrange basis for $\mathbb{Q}_k$ is $(l_{k,i_1}(x_1) \dots l_{k,i_n}(x_n))_{0 \leq i_1, \dots, i_n \leq k}$, where the $l_{k,i_j}(x_j)$ denote the 1D Lagrange basis functions at the interpolation points $x_{j,0}, \dots, x_{j,k}$. The dimension of $\mathbb{Q}_k$ in n dimensions is $(k+1)^n$, which is also the number of the basis functions. An example with $k = 2$ in 2D is given in Figure 1.2. Here the dimension of $\mathbb{Q}_2$ is nine and the nine basis functions are $l_{k,i}(x) l_{k,j}(y)$, $0 \leq i, j \leq 2$.

Discontinuous finite elements

Functions that are only in L^2 can be defined piecewise on each cell without any continuity requirements. Then there is no restriction on the degrees of freedom.

Classical discontinuous Finite Elements are *nodal* Lagrange elements based on Gauss points as the quadrature points. Even though their approximation order is lower for the same number of points it is sometimes convenient to place the degrees of freedom at Gauss-Lobatto points, where one point on each side is on the edge. This makes easier the computation of edge integrals. However in this case, as the Finite Element is discontinuous, degrees of freedom on the interface are not shared by neighbouring elements.

Another class of classical discontinuous Finite Elements are *modal* elements where the basis functions are generally taken to be the orthonormal Legendre polynomials $(L_i)_{0 \leq i \leq k}$, which have the advantage of yielding the identity as a mass matrix. The corresponding degrees of freedom are $\sigma_i(p) = \int p(x) L_i(x) \, \mathrm{d}x$. Such constructions, where each element of the basis has a deferent degree, are called hierarchical finite elements and facilitate what is called the p refinement consisting in obtaining a more accurate approximation by taking a higher order polynomial rather that refining the grid. Indeed in this case when refining only one basis function needs to be added to the existing one rather than replacing all the basis functions as would be necessary for the previously seen Finite Elements where all basis functions have the same degree.

1.5.3 $H(\mathbf{div}, \Omega)$ conforming Finite Elements

The $H(\mathrm{div}, \Omega)$ space consists of vector fields. Each element as n components in n dimensions. Moreover, as we have seen before $H(\mathrm{div}, \Omega)$ vector fields have a continuous normal component and a discontinuous tangential component. Let us explain how such elements can be constructed in two dimensions. This can be generalised to higher dimensions.

Tensor product construction. Due to the given continuity requirements a $H(\mathrm{div}, \Omega)$ conforming Finite Element can be constructed, by defining separately an approximation of the tangential component and of the normal component. In this case $K = [-1, 1] \times [-1, 1]$,

$$P = \{\mathbf{p} = (p_x, p_y)^\top \,|\, p_x \in \mathbb{Q}_{k-1,k},\ p_y \in \mathbb{Q}_{k,k-1}\},$$

where

$$\mathbb{Q}_{m,n} = \mathrm{Span}(x^\alpha y^\beta,\ 0 \leq \alpha \leq m, 0 \leq \beta \leq n).$$

For p_x the degrees of freedom are the Gauss points in x and the Gauss-Lobatto points in y, such that p_x is discontinuous in x and continuous in y. It is the other way for p_y. This enables arbitrarily high order $H(\mathrm{div}, \Omega)$ conforming elements.

The Raviart-Thomas (RT) Finite Element. K is a non degenerate triangle of vertices $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$. For the element of order $k+1$, $k \geq 0$, denoting by $\bar{\mathbb{P}}_k$ the space of homogeneous polynomials of degree k (*i.e.* all the monomials are exactly of degree k)

$$P = RT_k = (\mathbb{P}_k)^2 + \begin{pmatrix} x \\ y \end{pmatrix} \bar{\mathbb{P}}_k.$$

In two dimensions $\dim \mathbb{P}_k = (k+1)(k+2)/2$ and $\dim \bar{\mathbb{P}}_k = k+1$, so that $\dim RT_k = (k+1)(k+3)$. This yields in particular $\dim RT_0 = 3$, $\dim RT_1 = 8$, $\dim RT_2 = 15$. The degrees of freedom are constructed starting from the edges in order to enforce the continuity requirements. This is our first example of moment based degrees of freedom, defined by 1D moments along the edges to enforce continuity and then 2D moments in the triangle to complete the missing degrees of freedom.

$$\Sigma = \left\{ \mathbf{p} \mapsto \int_{e_i} \mathbf{p} \cdot \boldsymbol{\nu} s^l \,\mathrm{d}s, 0 \leq l \leq k, \quad \mathbf{p} \mapsto \int_K \mathbf{p}(x_1, x_2) \cdot x_i^l \,\mathrm{d}x_i, \quad 0 \leq l \leq k-1,\ i = 1, 2 \right\}.$$

We count here $k+1$ degrees of freedom for each of the three edges, and $2k$ inner degrees of freedom, which are in total $3(k+1)+2k(k+1)/2 = (k+1)(k+3)$ which is precisely the dimension of RT_k. To prove the unisolvence it is thus enough to prove that a polynomial which vanishes on all degrees of freedom is necessarily zero. Note that because the monomials in our degrees of freedom are a basis of the polynomial spaces $\mathbb{P}_k(e_i)$, $i = 1, 2, 3$ and $\mathbb{P}_{k-1}(K)$ respectively this is given by the following lemma:

Lemma 3 *Let $k \geq 0$, and $\mathbf{p} \in RT_k(K)$ then*

$$\int_{e_i} \mathbf{p} \cdot \boldsymbol{\nu}\, r \,\mathrm{d}s = 0, \quad \forall r \in \mathbb{P}_k(e_i), \tag{1.23}$$

$$\int_K \mathbf{p} \cdot \mathbf{q} \,\mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{q} \in (\mathbb{P}_{k-1})^2 \tag{1.24}$$

implies that $\mathbf{p} = 0$.

Remark 3 *Note that instead of the monomials one can use* (1.23) *and* (1.24) *applied to arbitrary bases of $\mathbb{P}_k(e_i)$ and $(\mathbb{P}_{k-1})^2$ respectively.*

Proof. Let $\mathbf{p} \in RT_k$ such that (1.23) and (1.24) are satisfied. Then $\mathbf{p} = \mathbf{q} + \begin{pmatrix} x \\ y \end{pmatrix} \psi$ with $\mathbf{q} \in (\mathbb{P}_k)^2$ and $\psi \in \bar{\mathbb{P}}_k$. Let us first observe that on any of the three edges e_i we have $\mathbf{p} \cdot \boldsymbol{\nu} \in \mathbb{P}_k(e_i)$, this is obvioulsy the case for $\mathbf{q} \cdot \boldsymbol{\nu}$. On the other hand if $(x(s), y(s))$ defines a parametrisation of e_i, then by definition of the normal $n_x(x(s) - x(s_0)) + n_y(y(s) - y(s_0)) = 0$ so that $n_x x(s) + n_y y(s)$ is constant on e_i. It follows that on the edge e_i

$$\boldsymbol{\nu} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \psi = (n_x x + n_y y)\psi.$$

Then as $(n_x x + n_y y)$ is a constant, $(n_x x + n_y y)\psi$ is a polynomial of degree k like ψ.

Now computing the divergence of $\mathbf{p}$ we find, as $\psi(x, y) = \sum_{l=0}^{k} \alpha_l x^l y^{k-l}$,

$$\nabla \cdot \mathbf{p} = \nabla \cdot \mathbf{q} + 2\psi + \begin{pmatrix} x \\ y \end{pmatrix} \cdot \nabla\psi = \nabla \cdot \mathbf{q} + (k+2)\psi.$$

Hence, taking $\mathbf{q} = \nabla\varphi$ with $\varphi \in \mathbb{P}_k$ in (1.24), and using Green's formula

$$\int_K \mathbf{p} \cdot \nabla\varphi \,\mathrm{d}\mathbf{x} = \int_{\partial K} \mathbf{p} \cdot \boldsymbol{\nu}\varphi \;\mathrm{d}s + \int_K \nabla \cdot \mathbf{p}\, \varphi \,\mathrm{d}\mathbf{x}.$$

we find, using (1.23) for the left hand side and (1.24) for the boundary term, that $\nabla \cdot \mathbf{p}$ is a polynomial of degree k orthogonal to all polynomials of degree k. Thus it is 0. Then because $\nabla \cdot \mathbf{q} \in \mathbb{P}_{k-1}$ it also follows that $\psi = 0$ and hence $\mathbf{p} \in (\mathbb{P}_k)^2$.

Now in order to conclude we assume that the reference triangle has one edge on $x = 0$ and one edge on $y = 0$ and is in the positive quarter plane. The the condition $\mathbf{p} \cdot \boldsymbol{\nu} = 0$ yields that $p_x = 0$ for $x = 0$, hence $p_x = x\psi_1$ with $\psi_1 \in \mathbb{P}_{k-1}$ and $p_y = 0$ for $y = 0$, hence $p_y = y\psi_2$ with $\psi_2 \in \mathbb{P}_{k-1}$. Then taking $\mathbf{q} = (\psi_1, \psi_2)^\top$ in (1.24), we obtain

$$\int_K (x\psi_1^2 + y\psi_2^2) \,\mathrm{d}\mathbf{x} = 0$$

from which it follows as both terms are positive that they are both zero. So $\mathbf{p} = 0$ which was what we needed to prove. ∎

Remark 4 *Obviously, the procedure used here to construct the Finite Element, by first setting the boundary degrees of freedom to ensure the required continuity and then complete with the needed inner degrees of freedom can also be applied to quads.*

The Brezzi-Douglas-Marini (BDM) Finite Element. K is a non degenerate triangle of vertices $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$. Here P is the full polynomial space in each direction $P = BDM_k = (\mathbb{P}_k)^2$. As continuity of the normal component is needed on each of the three faces, these requires at least three degrees of freedom. So and $H(\mathrm{div}, \Omega)$ conforming BDM_k element can only be constructed for $k \geq 1$ as $\dim BDM_0 = 2$. In order to define the degrees of freedom, we need the following space

$$N_k = (\mathbb{P}_k)^2 + \begin{pmatrix} y \\ -x \end{pmatrix} \bar{\mathbb{P}}_k,$$

This space has obviously the same dimension as the corresponding RT_k: $\dim N_k = \dim RT_k = (k+1)(k+3)$. We can now define the degrees of freedom for the BDM_k element: introducing $(\psi_j)_0 \leq j \leq k$ a basis of $\mathbb{P}_k$ on each edge (this could be the monomials s^j as we used for Raviart-Thomas or any other basis), and $(\boldsymbol{\varphi}_j)_{0 \leq (k-1)(k+1)}$ a basis of N_{k-2}

$$\Sigma = \left\{ \mathbf{p} \mapsto \int_{e_i} \mathbf{p} \cdot \boldsymbol{\nu} \psi_j \, \mathrm{d}s, 0 \leq j \leq k, \ i = 1, 2, 3, \right.$$
$$\left. \mathbf{p} \mapsto \int_K \mathbf{p} \cdot \boldsymbol{\varphi}_j \, \mathrm{d}\mathbf{x}, \quad 0 \leq j \leq (k-1)(k+1) \right\}.$$

We first notice, that the number of degrees of freedom is $3(k+1)+(k-1)(k+1) = (k+1)(k+2) = \dim P$. So that proving that $\mathbf{p} \in P$ must vanish if all these degrees of freedom are zero is enough to prove unisolvence. Similarly as for the Raviart-Thomas element this follows from the following:

Lemma 4 *Let $k \geq 1$, and $\mathbf{p} \in \mathbb{P}_k(K)^2$ then*

$$\int_{e_i} \mathbf{p} \cdot \boldsymbol{\nu} \, \psi \, \mathrm{d}s = 0, \quad \forall \psi \in \mathbb{P}_k(e_i), \tag{1.25}$$

$$\int_K \mathbf{p} \cdot \mathbf{q} \, \mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{q} \in N_{k-2} \tag{1.26}$$

implies that $\mathbf{p} = 0$.

Proof. Let $\mathbf{p} \in \mathbb{P}_k(K)^2$ verifying (1.25) and (1.26). For any $\varphi \in \mathbb{P}_{k-1}$, we have that $\nabla \varphi \in (\mathbb{P}_{k-2})^2 \subset N_{k-2}$, hence from (1.26) it follows that

$$\int_K \mathbf{p} \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} = 0 = -\int_K \nabla \cdot \mathbf{p} \, \varphi \, \mathrm{d}x + \int_{\partial K} \mathbf{p} \cdot \boldsymbol{\nu} \, \varphi \, \mathrm{d}s.$$

The last term is zero because of (1.25). Then $\nabla \cdot \mathbf{p}$ is a polynomial of degree $k-1$ orthogonal to all polynomials of degree $k-1$. Hence it is zero. Thus $\mathbf{p} = (\partial_y \psi, -\partial_x \psi)$ for a $\psi \in \mathbb{P}_{k+1}$. Moreover due to (1.25), $\mathbf{p} \cdot \boldsymbol{\nu} = 0$ on each edge, this implies that the tangential derivative of ψ on each edge, and so ψ is constant on all the edges. As it is defined up to a constant, on can choose ψ such that it vanishes on the three edges. Then introducing the barycentric coordinates

$\lambda_1, \lambda_2, \lambda_3$ of the triangle, it follows that each λ_i divides ψ, so that $\psi = \lambda_1\lambda_2\lambda_3\tilde{\psi}$ with $\tilde{\psi} \in \mathbb{P}_{k-2}$.

Let us now take $\mathbf{q} = \begin{pmatrix} y \\ -x \end{pmatrix} \tilde{\psi} \in N_{k-2}$. Then from (1.26) it follows that

$$\int_K \mathbf{p} \cdot \mathbf{q} \, \mathrm{d}\mathbf{x} = \int_K \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} \cdot \begin{pmatrix} y \\ -x \end{pmatrix} \tilde{\psi} \, \mathrm{d}\mathbf{x} = \int_K k\lambda_1\lambda_2\lambda_3\tilde{\psi}^2 \, \mathrm{d}x = 0$$

which implies that $\tilde{\psi} = 0$ and thus $\mathbf{p} = 0$, which is the desired result. ∎

Remark 5 *The extension to 3D is straightforward, the degrees of freedom stay the same, the triangles being replaced by tetrahedra and the edges being replaced by faces.*

1.5.4 $H(\mathbf{curl}, \Omega)$ conforming Finite Elements

Recall that for a 3D vector $\mathbf{u} = (u_1, u_2, u_3)^\top$, the curl is defined by the relation

$$\nabla \times \mathbf{u} = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}.$$

In 2D, there is no dependency on the third coordinate and the curl degenerates into the vector curl of a scalar, for the first two components and the scalar curl of a vector for the last component:

$$\nabla \times \mathbf{u} = \begin{pmatrix} \mathbf{curl}\, u_3 \\ \operatorname{curl} \underline{\mathbf{u}} \end{pmatrix} \quad \text{with } \mathbf{curl}\, u_3 = \begin{pmatrix} \partial_2 u_3 \\ -\partial_1 u_3 \end{pmatrix}, \ \underline{\mathbf{u}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ \operatorname{curl} \underline{\mathbf{u}} = \partial_1 u_2 - \partial_2 u_1.$$

$H(\mathrm{curl}, \Omega)$ conforming elements have a continuous tangential component. They can thus be simply obtained in 2D by exchanging the two components of the vector in the tensor product case.

For triangles, the natural finite element space, built as the Raviart-Thomas element is the Nédélec Element defined by $K = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, a non degenerate triangle,

$$P = N_k = (\mathbb{P}_k)^2 + \begin{pmatrix} y \\ -x \end{pmatrix} \bar{\mathbb{P}}_k,$$

and the degrees of freedom are obtained from the Raviart-Thomas degrees of freedom by replacing the normal $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$ by the tangent $\boldsymbol{\tau} = (\nu_2, -\nu_1)$. Unisolvence is also easily adapted from the Raviart-Thomas case and is based on the following

Lemma 5 *Let $k \geq 0$, and $\mathbf{p} \in RT_k(K)$ then*

$$\int_{e_i} \mathbf{p} \cdot \boldsymbol{\tau} \, r \, \mathrm{d}s = 0, \quad \forall r \in \mathbb{P}_k(e_i), \tag{1.27}$$

$$\int_K \mathbf{p} \cdot \mathbf{q} \, \mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{q} \in (\mathbb{P}_{k-1})^2 \tag{1.28}$$

implies that $\mathbf{p} = 0$.

Remark 6 *The extension to 3D for tensor product element is also quite natural by considering the components separetely and noticing that in 3D there are two tangential components and three normal components.*

Simplex based $H(curl, \Omega)$ Finite Elements, which are constructed on tetrahedra in 3D are fundamentally different from their $H(div, \Omega)$ counterpart, unlike in 2D where one could go from one to the other by a simple rotation of the components. There are also two classes of finite elements, like Raviart-Thomas and Brezzi-Douglas-Marini in 2D, which are called respectively Nédélec elements of first type and of second type.

The 3D Nédélec space in which the elements of first type live and which is used to define the degrees of freedom of the Nédélec elements of second type reads

$$N_k = (\mathbb{P}_k)^3 \oplus \left(\mathbf{x} \times (\mathbb{P}_k)^3\right) \tag{1.29}$$

and the degrees of freedom are defined by the following unisolvence lemma

Lemma 6 *Let K be a non degenerate tetrahedron, with faces $(f_i)_{0\leq i\leq 4}$, and edges $(e_i)_{0\leq i\leq 6}$ $k \geq 0$, and $\mathbf{p} \in N_k(K)$ then*

$$\int_{e_i} \mathbf{p} \cdot \boldsymbol{\tau}_i \, r \, \mathrm{d}s = 0, \quad \forall r \in \mathbb{P}_k(e_i), \tag{1.30}$$

$$\int_{f_i} (\mathbf{p} \times \boldsymbol{\nu}_i) \cdot \mathbf{s} \, \mathrm{d}s = 0, \quad \forall \mathbf{s} \in (\mathbb{P}_{k-1}(f_i))^2, \tag{1.31}$$

$$\int_K \mathbf{p} \cdot \mathbf{q} \, \mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{q} \in (\mathbb{P}_{k-2})^2 \tag{1.32}$$

implies that $\mathbf{p} = 0$.

We denote here by $\boldsymbol{\tau}_i$ the unit vector along the edge e_i and by $\boldsymbol{\nu}_i$ the outward unit normal on face f_i.

1.6 Change of local basis

The local degrees of freedom Σ of a Finite Element defined by (K, P, Σ) can also be used to compute the element matrices with respect to the matrices corresponding to a simpler or easily computable basis of the same space P of dimension N. This could for example be an orthonormal basis. This allows to compute the matrices once for all in some basis and then to get the matrices in any other basis by just computing the corresponding generalised Vandermode matrix as follows.

Let us denote by $(\phi_l)_{0\leq l\leq N}$ the reference basis in which the element matrices are known. Let now $(\hat{p}_j)_{0\leq j\leq N}$ be another basis of the same linear space P associated to the degrees of freedom $(\sigma_i)_{0\leq i\leq N}$. Then $\hat{p}_j$ can be expressed in the basis $(\phi_l)_{0\leq l\leq N}$:

$$\hat{p}_j(\mathbf{x}) = \sum_{l=1}^{N} \alpha_{j,l} \phi_l(\mathbf{x}). \tag{1.33}$$

Denoting by $\hat{\mathbf{p}} = (\hat{p}_1, \dots, \hat{p}_N)^\top$, $\boldsymbol{\phi} = (\phi_1, \dots, \phi_N)^\top$ and $A = ((\alpha_{j,l}))_{1 \le j,l \le N}$, this relation can be written in matrix form

$$\hat{\mathbf{p}}(\mathbf{x}) = A\boldsymbol{\phi}(\mathbf{x}).$$

Applying the linear forms σ_i to relation (1.33) for $1 \le i \le N$, we get

$$\mathbb{I}_N = VA^\top, \tag{1.34}$$

where $\mathbb{I}_N$ is the identity matrix and $V = ((\sigma_i(\phi_j)))_{1 \le i,j \le N}$. The matrix V is called generalised Vandermonde matrix, as in the case when $((\phi_j))_{1 \le j \le N}$ is the monomial basis $(1, x, \dots, x^{N-1})$ of $\mathbb{P}_k$ in 1D and for a Lagrange Finite Element where $\sigma_i(\hat{p}_j) = \hat{p}_j(x_i)$, the matrix V is the classical Vandermonde matrix $V = ((x_i^{j-1}))_{1 \le i,j \le N}$.

The generalised Vandermonde matrix V is explicitly computable when the basis $(\phi_j)_j$ and the degrees of freedom $(\sigma_i)_i$ are explicitly known. From relation (1.34), it follows immediately that $A = V^{-\top}$, where we denote by $V^{-\top}$ the inverse of the transpose of V.

One can also take the derivative of formula (1.33). Thus

$$\partial_x \hat{p}_j(\mathbf{x}) = \sum_{l=1}^{N} \alpha_{j,l} \partial_x \phi_l(\mathbf{x}),$$

and the same for the other variables if needed. Applying again the linear form σ_i to this equation for $1 \le i \le N$, we find

$$D_x \hat{\mathbf{p}} = D_x \boldsymbol{\phi} A^\top = D_x \boldsymbol{\phi} V^{-1},$$

where the matrices $D_x\hat{\mathbf{p}}$ and $D_x\boldsymbol{\phi}$ are defined respectively by their generic element $((\sigma_i(\partial_x \hat{p}_j)))_{1 \le i,j \le N}$ and $((\sigma_i(\partial_x \phi_j)))_{1 \le i,j \le N_k}$. Note that if $(\phi_j)_j$ has been chosen such $D_x\phi$ is explicitly computable, we can, thanks to this relation explicitly compute the matrix $D_x\hat{\mathbf{p}}$ and similarly the derivative matrices with respect to other variables.

We can express $\hat{M}$ the mass matrix on the reference element $\hat{K}$ using the Vandermonde matrix V. Using formula (1.33),

$$\hat{p}_j(\mathbf{x}) = \sum_{l=1}^{N} \alpha_{j,l} \phi_l(\mathbf{x}).$$

Hence

$$\int_{\hat{K}} \hat{p}_i(\mathbf{x}) \hat{p}_j(\mathbf{x})\, d\mathbf{x} = \sum_{l=1}^{N} \sum_{m=1}^{N} \alpha_{i,l} \alpha_{j,m} \int_{\hat{K}} \phi_l(\mathbf{x}) \phi_m(\mathbf{x})\, d\mathbf{x} = \sum_{l=1}^{N} \alpha_{i,l} \alpha_{j,m},$$

if the basis functions $(\phi_j)_j$ are orthonormal on $\hat{K}$. It follows that $\hat{M} = AA^\top = V^{-\top}V^{-1}$.

If the space P is stable by derivation, as is the case for the polynomial space $\mathbb{P}_k$, we can also express the derivative $\partial_x G_h$ or other derivatives in the basis

$(p_i)_i$ and this derivative can also be characterised by the degrees of freedom $(\sigma_i(\partial_x G_h))_{1\leq i\leq N_k}$. We denote by $D_x\mathbb{G}$ the vector containing these degrees of freedom, which can be expressed using $\mathbb{G}$ as follows. We have

$$\partial_x G_h(\mathbf{x}) = \sum_{j=1}^{N_k} \sigma_j(G_h)\partial_x p_j(\mathbf{x}),$$

and so for $1 \leq i \leq N$,

$$\sigma_i(\partial_x G_h) = \sum_{j=1}^{N_k} \sigma_j(G_h)\sigma_i(\partial_x p_j).$$

This can also be written $D_x\mathbb{G} = (D_x\mathbf{p})\mathbb{G}$, denoting by $D_x\mathbf{p}$ the matrix with generic term $((\sigma_i(\partial_x p_j)))_{1\leq i,j\leq N}$.

1.7 Construction of the Finite Element approximation space

1.7.1 The mapping

A conforming Finite Element space V_h, is constructed from a reference element, by mapping the reference Finite Element defined previously to a collection of physical elements that define an admissible mesh of the computational domain and ensuring that $V_h \subset V$ by equating degrees of freedom on the interface.

The coefficients of Finite Element matrices are defined as integrals involving the basis functions. The standard practice as well for convergence theory as for practical implementation is to make a change of variables from one unique reference element to each of the other elements. For Lagrange Finite Elements affine transformation can be used for simplices and multi-linear transformation for tensor product elements. For higher order elements, when one wants to get a corresponding high order approximation of the boundary, a polynomial transformation of the same order as the Finite Element approximation of the elements touching the boundary is used. This is called *isoparametric Finite Elements*.

Let us call $\hat{K} \subset \mathbb{R}^d$ the reference element and $K \subset \mathbb{R}^d$ any element of the physical mesh. We then define the mapping

$$\begin{aligned} \mathbf{F} : \hat{K} &\to K, \\ \hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_d)^\top &\mapsto \mathbf{x} = \mathbf{F}(\hat{\mathbf{x}}) = (F_1(\hat{\mathbf{x}})), \ldots, F_d(\hat{\mathbf{x}})))^\top. \end{aligned}$$

As differential operators are involved we will need the Jacobian matrix of the mapping, denoted by

$$DF(\hat{x}) = \left(\left(\frac{\partial F_i}{\partial \hat{x}_j}\right)\right)_{1\leq i,j\leq d}.$$

Let us denote by $J(\hat{x}) = \det(DF(\hat{x}))$, for any scalar function $v(\mathbf{x})$, $\hat{v}(\hat{\mathbf{x}}) = v(\mathbf{F}(\hat{\mathbf{x}}))$, by $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})^\top$ and finally by $\hat{\nabla} = (\frac{\partial}{\partial \hat{x}_1}, \ldots, \frac{\partial}{\partial \hat{x}_d})^\top$. We then find, using the chain rule that

$$\frac{\partial \hat{v}}{\partial \hat{x}_j} = \sum_{i=1}^{d} \frac{\partial F_i}{\partial \hat{x}_j} \frac{\partial v}{\partial x_i}.$$

So that

$$\hat{\nabla}\hat{v}(\hat{\mathbf{x}}) = (DF)^\top \nabla v(\mathbf{x}) \Rightarrow \nabla v(\mathbf{x}) = (DF)^{-\top} \hat{\nabla}\hat{v}(\hat{\mathbf{x}}). \tag{1.35}$$

An important special case is when $\mathbf{F}$ is an affine map. This is typically used for simplicial Finite Elements (except on the boundary for high order elements as it brings back the method to second order). The classical Finite Element theory is also based on affine mappings. In this case $\mathbf{F}(\hat{\mathbf{x}}) = A\hat{\mathbf{x}} + \mathbf{b}$, where A is a $d \times d$ matrix and $\mathbf{b}$ a d-component column vector. For a simplicial mesh, the affine transformation, that is the components of A and $\mathbf{b}$, is completely determined by the imposing that $\mathbf{F}(\hat{\mathbf{a}}_i) = \mathbf{a}_i$ $0 \leq i \leq d$, *i.e.* that all vertices of the reference element $\hat{K}$ are mapped to the vertices of the current element K. We then have, for $\hat{\mathbf{a}}_0 = (0, \ldots, 0)$ and the components of a_{ij} $(1 \leq j \leq d)$ of $\hat{\mathbf{a}}_i$ are all 0 except the i^{th} component which is 1,

$$A = \begin{pmatrix} a_{1,1} - a_{0,1} & a_{2,1} - a_{0,1} & \ldots & a_{d,1} - a_{0,1} \\ a_{1,2} - a_{0,2} & a_{2,2} - a_{0,2} & \ldots & a_{d,2} - a_{0,2} \\ \vdots & \vdots & \ldots & \vdots \\ a_{1,d} - a_{0,d} & a_{2,d} - a_{0,d} & \ldots & a_{d,d} - a_{0,d} \end{pmatrix}, \quad \mathbf{b} = \mathbf{a}_0.$$

The jacobian matrix of the affine transformation is then the matrix A.

1.7.2 Bases of tangent and cotangent spaces in curvilinear coordinates

A mapping from $\mathbf{F} : \mathbb{R}^d \to \mathbb{R}^d$ defines a curvilinear coordinate system, which induces a natural vector space called tangent space a each point. The column vectors of the jacobian matrix $D\mathbf{F}(\hat{\mathbf{x}})$, $\mathbf{t}_i = \frac{\partial \mathbf{F}}{\partial x_i}$ $1 \leq i \leq d$ define a basis of the tangent space, which is the image of the canonical basis under the tangent map defined by the Jacobian matrix $D\mathbf{F}(\hat{\mathbf{x}})$. In the case of an affine map $\mathbf{F}(\hat{\mathbf{x}}) = A\hat{\mathbf{x}}+\mathbf{b}$, mapping the reference d- dimensional simplex into the simplex $(\mathbf{a}_0, \ldots \mathbf{a}_d)$, we have $D\mathbf{F}(\hat{\mathbf{x}}) = A$, independently of $\hat{\mathbf{x}}$ and then $\mathbf{t}_i = \mathbf{a}_i - \mathbf{a}_0$, so that the tangent basis consists of the vectors going from $\mathbf{a}_0$ to each of the other vertices of the simplex. See Figure 1.3 for a sketch in a triangle.

Note that the direction and length of the tangent basis vectors $\mathbf{t}_i$ are completely determined by the mapping $\mathbf{F}$. They form a basis of $\mathbb{R}^d$ provided $D\mathbf{F}(\hat{\mathbf{x}})$ is invertible at all points, in other words, provided the mapping $\mathbf{F}$ is a C^1 diffeomorphism. However this basis is in general neither orthogonal nor normed. In an orthonormal basis the coefficients of any vector in this basis $\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_d\mathbf{e}_d$ can be expressed in the euclidian space $\mathbb{R}^d$, *i.e.* endowed with a scalar product, by $v_i = \mathbf{v} \cdot \mathbf{e}_i$. This does not work anymore in a non-orthogonal basis. This is why

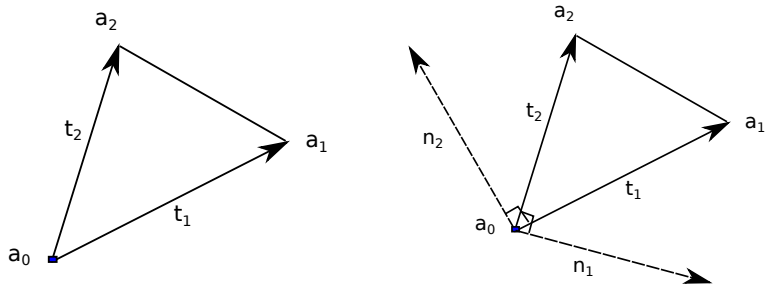

Figure 1.3: Left: Tangent basis, Right: Tangent and cotangent basis vectors

in this case it is useful to use the dual basis, which consists of elements of $\mathcal{L}(\mathbb{R}^d)$, that is linear forms on $\mathbb{R}^d$, which map any vector to a real number. The dual basis is by definition the unique basis of the dual space such that $d_i(\mathbf{t}_j) = \delta_{ij}$. Thanks to the scalar product, in a euclidian space, each element d_i of the dual basis can be uniquely associated to a vector $\mathbf{n}_i$ of $\mathbb{R}^d$ by defining $\mathbf{n}_i$ as the unique vector verifying

$$\mathbf{n}_i \cdot \mathbf{v} = d_i(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{R}^d.$$

Considering now the basis $(\mathbf{t}_1, \ldots, \mathbf{t}_d)$, the dual basis $\mathbf{n}_1, \ldots, \mathbf{n}_d \in \mathbf{R}^d$ is defined by $\mathbf{n}_i \cdot \mathbf{t}_j = \delta_{ij}$, which means that $\mathbf{n}_i$ is locally orthogonal to the hyperplane defined by $(\mathbf{t}_j)_{j \neq i}$ and normed by $\mathbf{n}_i \cdot \mathbf{t}_i = 1$.

For an affine mapping defined by a simplex, as introduced above, $\mathbf{n}_i$ is normal to the face defined by all the vertices but $\mathbf{a}_i$. See Figure 1.3 for a drawing in $\mathbb{R}^2$. An orthonormal basis is a special case: if the primal basis $(\mathbf{t}_1, \ldots, \mathbf{t}_d)$ is orthonormal, then we have, following the definition of the dual basis, that $\mathbf{n}_i = \mathbf{t}_i$ for $i = 1, \ldots, d$.

Lemma 7 *The matrix $N(\hat{\mathbf{x}}) = [\mathbf{n}_1, \ldots, \mathbf{n}_d]$, whose columns are the elements of the dual basis of $(\mathbf{t}_1, \ldots, \mathbf{t}_d)$ verifies*

$$N(\hat{\mathbf{x}}) = (D\mathbf{F}(\hat{\mathbf{x}}))^{-\top}. \tag{1.36}$$

Proof. Let us denote $D\mathbf{F}(\hat{\mathbf{x}}) = [\mathbf{t}_1, \ldots, \mathbf{t}_d]$ to highlight that the tangent vectors are the columns of the Jacobian matrix. Then, due to the definition of the dual basis, we notice that

$$N^\top D\mathbf{F} = ((\mathbf{n}_i \cdot \mathbf{t}_j))_{1 \leq i,j \leq d} = ((\delta_{ij}))_{1 \leq i,j \leq d} = \mathbb{I}_d.$$

This shows that $N^\top = (D\mathbf{F})^{-1}$, which is equivalent to the result we are looking for.

■

Proposition 1 *In a three dimensional space ($d = 3$), the normal vectors are obtained from the tangent vector through the following relation:*

$$\mathbf{n}_1 = \frac{1}{J}\mathbf{t}_2 \times \mathbf{t}_3, \quad \mathbf{n}_2 = \frac{1}{J}\mathbf{t}_3 \times \mathbf{t}_1, \quad \mathbf{n}_3 = \frac{1}{J}\mathbf{t}_1 \times \mathbf{t}_2.$$

Proof. As $\mathbf{n}_1$ is by definition orthogonal to $\mathbf{t}_2$ and $\mathbf{t}_3$ is is colinear to $\mathbf{t}_2 \times \mathbf{t}_3$ and can be written $\mathbf{n}_1 = \alpha \mathbf{t}_2 \times \mathbf{t}_3$ for some scalar α. Now as $\mathbf{t}_1 \cdot \mathbf{n}_1 = 1$ and $\mathbf{t}_1 \cdot \mathbf{t}_2 \times \mathbf{t}_3 = J$, we find $\alpha = 1/J$. The same reasoning applies to the two other components. ∎

Remark 7 *In the physics literature, the tangent basis is generally called* covariant *basis and the basis of the dual cotangent space is called* contravariant *basis.*

1.7.3 H^1 conforming Finite Elements

Let us build the conforming Finite Element space V_h^0 based on the reference element $(\hat{K}, \hat{P}, \hat{\Sigma})$ as a subspace of $H^1(\Omega)$. For this we need a mesh denoted by $\mathcal{T} = \bigcup_{1 \le e \le N_{el}} K_i$ consisting of N_{el} disjoint conforming elements denoted each by K_i. On this mesh we can then define $V_h^0 \subset H^1(\Omega)$ by

$$V_h^0 = \{v_h \in C^0(\Omega) \,|\, v_{h|K_e} = \hat{v}_e \circ F_e^{-1},\ \hat{v}_e \in \hat{P},\quad 1 \le e \le N_{el}\}.$$

This means that the space V_h^0 is defined element by element and that the finite dimensional space on each element is defined as composition of the reference element space $\hat{P}$ and the local element mapping F_e. In the case when F_e is an affine mapping and $\hat{P}$ is a polynomial space, the mapped space is still a polynomial space of the same degree, but not in the general case. Let us denote by $N = \dim V_h^0$ the dimension of the full Finite Element approximation space, and $\hat{N} = \dim \hat{P}$ the dimension of the local approximation space on each Element. If there are degrees of freedom shared between several elements we have $N < N_{el}\hat{N}$, and in all cases $N \le N_{el}\hat{N}$.

The Finite Element space V_h^0 being defined, the next step is to restrict the variational formulation of the problem being considered to functions in V_h^0. For the Poisson problem with Neumann boundary conditions (1.18) this leads to the following discrete variational formulation:
Find $u_h \in V_h^0$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \,\mathrm{d}\mathbf{x} + \int_\Omega u_h v_h \,\mathrm{d}\mathbf{x} = \int_\Omega f v_h \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} g v_h \,\mathrm{d}\sigma \quad \forall v_h \in V_h^0. \tag{1.37}$$

This discrete variational formulation now this to be transformed to a linear system for practical implementation on a computer. To this aim, the integrals need to be decomposed on single elements and then mapped back to the reference elements so that the basis functions of the reference Finite Element space $\hat{P}$ can be plugged in.

First splitting the integrals over the elements (1.37) becomes

$$\sum_{e=1}^{N_{el}} \int_{K_e} \nabla u_h \cdot \nabla v_h \,\mathrm{d}\mathbf{x} + \sum_{e=1}^{N_{el}} \int_{K_e} u_h v_h \,\mathrm{d}\mathbf{x} = \sum_{e=1}^{N_{el}} \int_{K_e} f v_h \,\mathrm{d}\mathbf{x} + \sum_{e=1}^{N_{bel}} \int_{\Gamma_e} g v_h \,\mathrm{d}\sigma \quad \forall v_h \in V_h^0.$$

The last sum is on the element touching the boundary of Ω. The next step is to make the change of variables $\mathbf{x} = F_e(\hat{\mathbf{x}})$ in the integral over K_i. Then $\mathrm{d}\mathbf{x} =$

$J(\hat{\mathbf{x}})\,\mathrm{d}\hat{\mathbf{x}}$, where as introduced above, $J = \det DF$. In the integrals not involving derivatives we then simply get

$$\int_{K_e} u_h v_h \,\mathrm{d}\mathbf{x} = \int_{\hat{K}} \hat{u}_e \hat{v}_e J \,\mathrm{d}\hat{\mathbf{x}}, \qquad \int_{K_e} f v_h \,\mathrm{d}\mathbf{x} = \int_{\hat{K}} f \hat{v}_e J \,\mathrm{d}\hat{\mathbf{x}}.$$

The boundary integral is dealt with the same way, using the restriction of the element mappings to the boundary and the appropriate Jacobian in the change of variables. For the integral involving the gradients, we need to make use of the chain rule (1.35)

$$\int_{K_e} \nabla u_h \cdot \nabla v_h \,\mathrm{d}\mathbf{x} = \int_{\hat{K}} (DF)^{-\top} \hat{\nabla} \hat{u}_e \cdot (DF)^{-\top} \hat{\nabla} \hat{v}_e J \,\mathrm{d}\hat{\mathbf{x}}.$$

Finally, we can express $\hat{u}_e, \hat{v}_e \in \hat{P}$, on the basis of $\hat{P}$ corresponding to the degrees of freedom $\hat{\Sigma}$. Let us denote by $(\hat{\varphi}_i)_{1 \le i \le \hat{N}}$ this basis. Then

$$\hat{u}_e(\hat{\mathbf{x}}) = \sum_{j=1}^{\hat{N}} u_j \hat{\varphi}_j(\hat{\mathbf{x}}), \qquad \hat{v}_e(\hat{\mathbf{x}}) = \sum_{i=1}^{\hat{N}} v_i \hat{\varphi}_i(\hat{\mathbf{x}}).$$

By plugging these expressions in the element integrals defining the variational formulation we obtain a linear system. First

$$\int_{\hat{K}} \hat{u}_e \hat{v}_e J \,\mathrm{d}\hat{\mathbf{x}} = \sum_{1 \le i,j \le \hat{N}} u_j v_i \int_{\hat{K}} \hat{\varphi}_i \hat{\varphi}_j J \,\mathrm{d}\hat{\mathbf{x}} = V^\top \hat{M}_e U, \text{ with } \hat{M}_e = ((\int_{\hat{K}} \hat{\varphi}_i \hat{\varphi}_j J \,\mathrm{d}\hat{\mathbf{x}}))_{1 \le i,j \le \hat{N}}$$

where $\hat{M}_e$ is called the element mass matrix, $V = (v_1, \ldots, v_{\hat{N}})^\top$, $U = (u_1, \ldots, u_{\hat{N}})^\top$. In the same way

$$\int_{\hat{K}} (DF)^{-\top} \hat{\nabla} \hat{u}_e \cdot (DF)^{-\top} \hat{\nabla} \hat{v}_e J \,\mathrm{d}\hat{\mathbf{x}} = V^\top \hat{A}_e U, \text{ with } \hat{A}_e = \int_{\hat{K}} (DF)^{-\top} \hat{\nabla} \hat{\varphi}_i \cdot (DF)^{-\top} \hat{\nabla} \hat{\varphi}_j J \,\mathrm{d}\hat{\mathbf{x}}.$$

$\hat{A}_e$ is called the element *stiffness matrix*. Assuming for simplicity that the boundary term g vanishes, it remains to assemble the right-hand-side depending on f. The contribution on the element K_e is obtained by

$$\int_{\hat{K}} f \hat{v}_e J \,\mathrm{d}\hat{\mathbf{x}} = \sum_{i=1}^{\hat{N}} v_i \int_{\hat{K}} f \hat{\varphi}_i J \,\mathrm{d}\hat{\mathbf{x}} = \hat{V}^\top b, \qquad \text{with } b = ((\int_{\hat{K}} f \hat{\varphi}_i J \,\mathrm{d}\hat{\mathbf{x}}))_{1 \le i \le \hat{N}}.$$

From local to global. To go from the element matrices to the global matrices, the sum over the elements need to be performed and contributions of degrees of freedom shared between different elements need to be added. For this a data structure associating a global number for each degree of freedom from the local number in each element needs to be created. This can define by a function Ψ which to a couple, (element, local degree of freedom) (e, i) associates the global number

of the degree of freedom I ($1 \leq I \leq dim(V_h^0)$) of the basis function φ_i of V_h^0 such that $(\varphi_I)_{|K_e} = \hat{\varphi}_i$. Then the global matrix A is obtained by the algorithm:

A=0;
for $e = 1, N_{el}$ **do**
  for $i, j = 1, \hat{N}$ **do**
    $$A(\Psi(e,i), \Psi(e,j)) = A(\Psi(e,i), \Psi(e,j)) + \hat{A}_e(i,j),$$
  end
end

where $\hat{A}_e$ is the element stiffness matrix on K_e. The same procedure applies to all matrices involved in the variational formulation and also to the right-hand-side.

1.7.4 $H(\mathbf{curl}, \Omega)$ conforming Finite Elements

Let us consider here the 3D case in which the curl operator is naturally defined. The 2D case can be obtained as a special case. The $H(\mathrm{curl}, \Omega)$ function space and its conforming approximation consists of vectors. Hence on a change of basis, one needs to decide how the components of the vectors are transformed. In order to have a natural map of the tangents to a face used in the definition of the degrees of freedom and that guarantee conformity, the vectors defining the edges of the reference element needed to be mapped to either the covariant or the contravariant basis associated to the mapping. In the $H(\mathrm{curl}, \Omega)$ case it turns out that the right basis is the contrariant basis $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$.

A vector $\mathbf{v}$ on the image element is defined from a vector $\hat{\mathbf{v}}$ on the reference element by the transformation

$$\mathbf{v}(\mathbf{x}) = (D\mathbf{F}(\hat{\mathbf{x}}))^{-\top} \hat{\mathbf{v}}(\hat{\mathbf{x}}), \quad \text{with } \mathbf{x} = \mathbf{F}(\hat{\mathbf{x}}). \tag{1.38}$$

This transformation is called the covariant Piola transform. In the context of differential forms it is the natural pullback transformation of a 1-form (in a 3D space).

As the columns of $(DF(\hat{\mathbf{x}}))^{-\top}$ are $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ as observed in (1.36), we see with this transformation formula that

$$\mathbf{v}(\mathbf{x}) = \hat{v}_1(\hat{\mathbf{x}})\mathbf{n}_1(\hat{\mathbf{x}}) + \hat{v}_2(\hat{\mathbf{x}})\mathbf{n}_2(\hat{\mathbf{x}}) + \hat{v}_3(\hat{\mathbf{x}})\mathbf{n}_3(\hat{\mathbf{x}}), \tag{1.39}$$

and in particular that the canonical basis $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ on the reference element is mapped into $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$. We will also need the images of the outward normals to the faces of the reference elements by this mapping.

Lemma 8 *The image of any outward normal $\hat{\boldsymbol{\nu}}$ of the reference element by the mapping* (1.38) *is $\lambda\boldsymbol{\nu}$, with $\lambda = \|\mathbf{n}_i\|$ in case the face of the referece element is orthogonal to $\hat{\mathbf{e}}_i$ and for the face opposite to the origin in the case of a tetrahedron $\lambda = \|\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3)\|/\sqrt{3}$.*

Proof. The unit outward normal is constant on each face of the reference element.

For a tetrahedral reference element the unit normal to the face opposite to $\mathbf{a}_1$ is $\hat{\boldsymbol{\nu}}_1 = \hat{\mathbf{e}}_1 = (1,0,0)^\top$, to the face opposite to $\mathbf{a}_2$ is $\hat{\boldsymbol{\nu}}_2 = \hat{\mathbf{e}}_2 = (0,1,0)^\top$ and to the face opposite to $\mathbf{a}_3$ is $\hat{\boldsymbol{\nu}}_3 = \hat{\mathbf{e}}_3 = (0,0,1)^\top$. Hence $\hat{\boldsymbol{\nu}}_i$ is mapped into $\mathbf{n}_i$ the corresponding vector of the contravariant basis. Because the outward unit vector on each face is orthogonal to the face, $\mathbf{n}_i$ and $\boldsymbol{\nu}$ are aligned and in the same direction. So the scaling factor with respect to the unit normal $\boldsymbol{\tau}$, whose norm is one, is given by $\|\mathbf{n}_i\|$. On the face opposite to $\hat{\mathbf{a}}_0$, the outward unit normal is $\hat{\boldsymbol{\nu}} = \frac{1}{\sqrt{3}}(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)$, which is mapped by the transformation to $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3)$. On the other hand, this $\mathbf{n}$ is orthogonal to the face $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ of the element as $\mathbf{n}\cdot(\mathbf{t}_2 - \mathbf{t}_1) = \mathbf{n}\cdot(\mathbf{t}_3 - \mathbf{t}_1) = 0$. Here the scaling factor compared to the unit normal is $\|\mathbf{n}\|$.

In the case of a cubic reference element, all the normals are parallel to one of the vector $\hat{\mathbf{e}}_i$ and the first part of the tetrahedron case applies.

■

In order to get an expression of a variational formulation defined in $H(\mathrm{curl}, \Omega)$, we now need to compute the curl of a vector in this basis.

Proposition 2 *We have the following properties:*

1. *For any vector field* $\mathbf{v}$*, defining* $D\mathbf{v}$ *the Jacobian matrix of* $\mathbf{v}$*, we have*

$$(\nabla \times \mathbf{v}) \times \mathbf{w} = \left((D\mathbf{v})^T - D\mathbf{v}\right)\mathbf{w} \quad \forall \mathbf{w} \in \mathbb{R}^d.$$

2. *The transformation rule of the Jacobian matrix under the transformation* (1.38) *reads*

$$D\mathbf{v}(\mathbf{x}) = S(\hat{\mathbf{x}}) + (D\mathbf{F}(\hat{\mathbf{x}}))^{-\top}\hat{D}\hat{\mathbf{v}}(\hat{\mathbf{x}})(D\mathbf{F}(\hat{\mathbf{x}}))^{-1}$$

   where S *is a symmetric matrix.*

3. *The transformation rule for the curl under the transformation* (1.38) *reads*

$$\nabla \times \mathbf{v}(\mathbf{x}) = \frac{1}{J(\hat{\mathbf{x}})} D\mathbf{F}(\hat{\mathbf{x}})\hat{\nabla} \times \hat{\mathbf{v}}(\hat{\mathbf{x}}). \tag{1.40}$$

Proof. 1. Can be checked by comparing the expressions on the left and right hand sides.

2. We can express(1.38) as

$$\hat{\mathbf{v}}(\hat{\mathbf{x}}) = (D\mathbf{F})^\top(\hat{\mathbf{x}})\mathbf{v}(\mathbf{F}(\hat{\mathbf{x}})),$$

so that

$$\hat{v}_i(\hat{\mathbf{x}}) = \sum_{k=1}^{3} \frac{\partial F_k(\hat{\mathbf{x}})}{\partial \hat{x}_i} v_k(\mathbf{F}(\hat{\mathbf{x}})).$$

Let us now compute the coefficient on line i and column j of $D\hat{\mathbf{v}}(\hat{\mathbf{x}})$.

$$\frac{\partial \hat{v}_i}{\partial \hat{x}_j} = \sum_{k=1}^{3} \left(\frac{\partial^2 F_k(\hat{\mathbf{x}})}{\partial \hat{x}_i \partial \hat{x}_j} v_k(\mathbf{F}(\hat{\mathbf{x}})) \right) + \sum_{k=1}^{3} \left(\frac{\partial F_k(\hat{\mathbf{x}})}{\partial \hat{x}_i} \sum_{l=1}^{3} \frac{\partial F_l}{\partial \hat{x}_j} \frac{\partial v_k}{\partial x_l} \right),$$

so that we recognise that

$$\hat{D}\hat{\mathbf{v}} = \tilde{S} + (D\mathbf{F})^\top (D\mathbf{v}(\mathbf{x}))(D\mathbf{F}),$$

where $\tilde{S} = \sum_{k=1}^{3} \left(\frac{\partial^2 F_k(\hat{\mathbf{x}})}{\partial \hat{x}_i \partial \hat{x}_j} v_k(\mathbf{F}(\hat{\mathbf{x}})) \right)$ is obviously symmetric. Multiplying this expression by $(D\mathbf{F})^{-\top}$ on the left and by $(D\mathbf{F})^{-1}$ on the right yields the result.

3. Using 1. and 2. we get, for an arbitrary vector $\mathbf{w} \in \mathbb{R}^3$ we find

$$(\nabla \times \mathbf{v}) \times \mathbf{w} = (D\mathbf{F})^{-\top} \left((\hat{D}\hat{\mathbf{v}})^\top - \hat{D}\hat{\mathbf{v}} \right) (D\mathbf{F})^{-1}\mathbf{w} = (D\mathbf{F})^{-\top} \left((\hat{\nabla} \times \hat{\mathbf{v}}) \times (D\mathbf{F})^{-1}\mathbf{w} \right). \tag{1.41}$$

Not denoting by $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ the columns of the matrix $(D\mathbf{F})^{-1}$ and taking $\mathbf{w} = \mathbf{e}_1 = (1,0,0)^\top$ so that $(D\mathbf{F})^{-1}\mathbf{w} = \mathbf{c}_1$, we notice that

$$(\nabla \times \mathbf{v}) \times \mathbf{e}_1 = \begin{pmatrix} 0 \\ (\nabla \times \mathbf{v})_3 \\ -(\nabla \times \mathbf{v})_2 \end{pmatrix} = \begin{pmatrix} \mathbf{c}_1 \cdot (\hat{\nabla} \times \hat{\mathbf{v}}) \times \mathbf{c}_1 \\ \mathbf{c}_2 \cdot (\hat{\nabla} \times \hat{\mathbf{v}}) \times \mathbf{c}_1 \\ \mathbf{c}_3 \cdot (\hat{\nabla} \times \hat{\mathbf{v}}) \times \mathbf{c}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{c}_1 \times \mathbf{c}_2 \cdot (\hat{\nabla} \times \hat{\mathbf{v}}) \\ -\mathbf{c}_3 \times \mathbf{c}_1 \cdot (\hat{\nabla} \times \hat{\mathbf{v}}) \end{pmatrix}.$$

And in the same way taking $\mathbf{w} = \mathbf{e}_2$ the second vector of the canonical basis

$$(\nabla \times \mathbf{v}) \times \mathbf{e}_2 = \begin{pmatrix} -(\nabla \times \mathbf{v})_3 \\ 0 \\ (\nabla \times \mathbf{v})_1 \end{pmatrix} = \begin{pmatrix} \mathbf{c}_1 \cdot (\hat{\nabla} \times \hat{\mathbf{v}}) \times \mathbf{c}_2 \\ \mathbf{c}_2 \cdot (\hat{\nabla} \times \hat{\mathbf{v}}) \times \mathbf{c}_2 \\ \mathbf{c}_3 \cdot (\hat{\nabla} \times \hat{\mathbf{v}}) \times \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} -\mathbf{c}_1 \times \mathbf{c}_2 \cdot (\hat{\nabla} \times \hat{\mathbf{v}}) \\ 0 \\ \mathbf{c}_2 \times \mathbf{c}_3 \cdot (\hat{\nabla} \times \hat{\mathbf{v}}) \end{pmatrix}.$$

It now remains to ckeck that the inverse $D\mathbf{F}$ of $(D\mathbf{F})^{-1}$ is defined by its three lines $(\mathbf{c}_2 \times \mathbf{c}_3)/J$, $(\mathbf{c}_3 \times \mathbf{c}_1)/J$, $(\mathbf{c}_1 \times \mathbf{c}_2)/J$ to realise that the three components of the curl indeed satisfy (1.40). ∎

Now using transformation (1.38) one can check by direct computation that all terms appearing in the curl Green's formula (1.8) including the normal component are exactly preserved.

Proposition 3 *We have the following properties:*

(i)

$$\int_K \nabla \times \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} = \int_{\hat{K}} \hat{\nabla} \times \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} \, \mathrm{d}\hat{\mathbf{x}} \tag{1.42}$$

(ii) For any face f_i on the boundary of K image of the face $\hat{f}_i$ of the reference element

$$\int_{f_i} \mathbf{u} \times \boldsymbol{\nu} \cdot \mathbf{v} \, \mathrm{d}\boldsymbol{\sigma} = \int_{\hat{f}_i} \hat{\mathbf{u}} \times \hat{\boldsymbol{\nu}} \cdot \hat{\mathbf{v}} \, \mathrm{d}\hat{\boldsymbol{\sigma}} \tag{1.43}$$

We observe in particular, that the term involving the curl is completely independent of the metric and has the same expression under any mapping. This is true also for the boundary term.

Proof. (i) Let us observe, as the columns of $D\mathbf{F}$ are the vectors $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ that due to (1.40)

$$\nabla \times \mathbf{u}(\mathbf{x}) = \frac{1}{J(\hat{\mathbf{x}})} \left(c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2 + c_3 \mathbf{t}_3 \right),$$

where $c_1 = \partial_{\hat{x}_2}\hat{u}_3 - \partial_{\hat{x}_3}\hat{u}_2$, $c_2 = \partial_{\hat{x}_3}\hat{u}_1 - \partial_{\hat{x}_1}\hat{u}_3$, $c_3 = \partial_{\hat{x}_1}\hat{u}_2 - \partial_{\hat{x}_2}\hat{u}_1$. Then, making the change of variables $\mathbf{x} = \mathbf{F}(\hat{\mathbf{x}})$, $\mathrm{d}\mathbf{x} = J(\hat{\mathbf{x}})\,\mathrm{d}\hat{\mathbf{x}}$ and plugging this expression into (1.42), we find

$$\begin{aligned}\int_K \nabla \times \mathbf{u} \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} &= \int_{\hat{K}} \frac{1}{J(\hat{\mathbf{x}})}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2 + c_3\mathbf{t}_3)\cdot(\hat{v}_1\mathbf{n}_1 + \hat{v}_2\mathbf{n}_2 + \hat{v}_3\mathbf{n}_3)J(\hat{\mathbf{x}})\,\mathrm{d}\hat{\mathbf{x}} \\ &= \int_{\hat{K}} (c_1\hat{v}_1 + c_2\hat{v}_2 + c_3\hat{v}_3)\,\mathrm{d}\hat{\mathbf{x}} = \int_{\hat{K}} \hat{\nabla} \times \hat{\mathbf{u}} \cdot \hat{\mathbf{v}}\,\mathrm{d}\hat{\mathbf{x}}.\end{aligned}$$

(ii) Lemma 8 gives us the image of the reference unit normal by the mapping. Let us first consider the case of a face of the reference element parallel to a coordinate plane. Let us express $\mathbf{u}$ and $\mathbf{v}$ in the contravariant basis using formula (1.39) and plug it into the expression of the left-hand side of (1.43), considering the face orthogonal to $\mathbf{n}_1$, the others being dealt with in the same way. Note also that the boundary is defined as a parametric surface for which the area measure is defined by $\mathrm{d}\boldsymbol{\sigma} = \|\mathbf{t}_2 \times \mathbf{t}_3\|\,\mathrm{d}\hat{\boldsymbol{\sigma}}$, where $\mathrm{d}\hat{\boldsymbol{\sigma}} = \mathrm{d}\hat{x}_2\,\mathrm{d}\hat{x}_3$. We use also that by definition of $\mathbf{n}_1$ we have $\|\mathbf{t}_2 \times \mathbf{t}_3\| = J\|n_1\|$. We introduce the parameter s to indicate whether $\mathbf{n}_1$ and ν are in the same direction or not.

$$\begin{aligned}\int_{f_i} \mathbf{u} \times \boldsymbol{\nu} \cdot \mathbf{v}\,\mathrm{d}\boldsymbol{\sigma} &= \int_{\hat{f}_i} (\hat{u}_1(\hat{\mathbf{x}})\mathbf{n}_1(\hat{\mathbf{x}}) + \hat{u}_2(\hat{\mathbf{x}})\mathbf{n}_2(\hat{\mathbf{x}}) + \hat{u}_3(\hat{\mathbf{x}})\mathbf{n}_3(\hat{\mathbf{x}})) \times (s\mathbf{n}_1/\|\mathbf{n}_1\|) \\ &\qquad \cdot (\hat{v}_1(\hat{\mathbf{x}})\mathbf{n}_1(\hat{\mathbf{x}}) + \hat{v}_2(\hat{\mathbf{x}})\mathbf{n}_2(\hat{\mathbf{x}}) + \hat{v}_3(\hat{\mathbf{x}})\mathbf{n}_3(\hat{\mathbf{x}}))J\|\mathbf{n}_1\|\,\mathrm{d}\hat{x}_2\,\mathrm{d}\hat{x}_3 \\ &= s\int_{\hat{f}_i} (\hat{u}_3\hat{v}_2 - \hat{u}_2\hat{v}_3)(\mathbf{n}_3 \times \mathbf{n}_1 \cdot \mathbf{n}_2)J\,\mathrm{d}\hat{x}_2\,\mathrm{d}\hat{x}_3 = s\int_{\hat{f}_i} (\hat{u}_3\hat{v}_2 - \hat{u}_2\hat{v}_3)\,\mathrm{d}\hat{x}_2\,\mathrm{d}\hat{x}_3 \\ &= \int_{\hat{f}_i} \hat{\mathbf{u}} \times \hat{\boldsymbol{\nu}}_1 \cdot \hat{\mathbf{v}}\,\mathrm{d}\hat{\boldsymbol{\sigma}}\end{aligned}$$

as $\mathbf{n}_3 \times \mathbf{n}_1 \cdot \mathbf{n}_2 = 1/J$ ∎

We can now assemble the matrices involved in a variational formulation posed on $H(\mathrm{curl}, \Omega)$. Let us consider for example the variational formulation
Find $\mathbf{u} \in H(curl, \Omega)$ *such that*

$$\int_\Omega \mathbf{u} \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Omega \nabla \times\ \mathbf{u} \cdot \nabla \times\ \mathbf{v}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in H(\mathrm{curl}, \Omega). \tag{1.44}$$

Let us build the conforming Finite Element space $V_h^1 \subset H(\mathrm{curl}, \Omega)$ based on the reference element $(\hat{K}, \hat{P}^1, \hat{\Sigma}^1)$. For this we need a mesh denoted by $\mathcal{T} = \bigcup_{1 \le e \le N_{el}} K_i$ consisting of N_{el} disjoint conforming elements denoted each by K_i. On this mesh we can then define $V_h^1 \subset H(\mathrm{curl}, \Omega)$ by

$$V_h^1 = \{\mathbf{v}_h \in H(\mathrm{curl}, \Omega)\,|\,\mathbf{v}_{h|K_e} = (D\mathbf{F})^{-\top}\hat{\mathbf{v}}_e \circ F_e^{-1},\ \hat{\mathbf{v}}_e \in \hat{P}^1,\ \ 1 \le e \le N_{el}\}.$$

Now we can in the variational formulation (1.44) replace $H(\mathrm{curl}, \Omega)$ by V_h^1 and $\mathbf{u}$ and $\mathbf{v}$ by their discrete counterparts

$$\mathbf{u}_h(\mathbf{x}) = \sum_{j=1}^{N_1} u_j \boldsymbol{\varphi}_j^1(\mathbf{x}), \qquad \mathbf{v}_h(\mathbf{x}) = \sum_{i=1}^{N_1} v_i \boldsymbol{\varphi}_i^1(\mathbf{x}),$$

where $(\boldsymbol{\varphi}_i^1)_{1\leq i\leq N_1}$ stands for the basis of V_h^1 associated to the Finite Element. As for the $H^1(\Omega)$ conforming case, plugging these expressions into the variational formulation leads to the matrix expression:

$$V^\top M_1 U + V^\top A_1 U = V^\top \mathbf{b},$$

where $M_1 = ((\int_\Omega \boldsymbol{\varphi}_i^1 \cdot \boldsymbol{\varphi}_j^1 \,\mathrm{d}\mathbf{x}))_{i,j}$ is the mass matrix in V_h^1, $A_1 = ((\int_\Omega \nabla\times \boldsymbol{\varphi}_i^1 \cdot \nabla \times \boldsymbol{\varphi}_j^1 \,\mathrm{d}\mathbf{x}))_{i,j}$, and $b = (\int \mathbf{f}\cdot\boldsymbol{\varphi}_i^1)_i$. There now only remains to compute these integrals. Let us start with the mass matrix M_1.

$$\int_\Omega \boldsymbol{\varphi}_i^1 \cdot \boldsymbol{\varphi}_j^1 \,\mathrm{d}\mathbf{x} = \sum_{e=1}^{N_{el}} \int_{K_e} \boldsymbol{\varphi}_i^1 \cdot \boldsymbol{\varphi}_j^1 \,\mathrm{d}\mathbf{x}.$$

Then on each element K_e we make the change of variables $\mathbf{x} = \mathbf{F}_e(\hat{\mathbf{x}})$, $\mathrm{d}\mathbf{x} = J_e(\hat{\mathbf{x}})\,\mathrm{d}\hat{\mathbf{x}}$ to get

$$\begin{aligned}
\int_{K_e} \boldsymbol{\varphi}_i^1 \cdot \boldsymbol{\varphi}_j^1 \,\mathrm{d}\mathbf{x} &= \int_{\hat{K}} (D\mathbf{F}_e)^{-\top}\hat{\boldsymbol{\varphi}}_i^1 \cdot (D\mathbf{F}_e)^{-\top}\hat{\boldsymbol{\varphi}}_j^1 J_e \,\mathrm{d}\hat{\mathbf{x}},\\
&= \int_{\hat{K}} (\hat{\varphi}_{i,1}^1\mathbf{n}_1 + \hat{\varphi}_{i,2}^1\mathbf{n}_2 + \hat{\varphi}_{i,3}^1\mathbf{n}_3)\cdot(\hat{\varphi}_{j,1}^1\mathbf{n}_1 + \hat{\varphi}_{j,2}^1\mathbf{n}_2 + \hat{\varphi}_{j,3}^1\mathbf{n}_3) J_e \,\mathrm{d}\hat{\mathbf{x}},\\
&= \int_{\hat{K}} (\hat{\boldsymbol{\varphi}}_i^1)^\top G_n^e \hat{\boldsymbol{\varphi}}_j^1 J_e \,\mathrm{d}\hat{\mathbf{x}},
\end{aligned}$$

where $G_n^e = ((\mathbf{n}_i\cdot\mathbf{n}_j))_{1\leq i,j\leq 3}$ is the contravariant metric tensor. This is the element mass matrix on the element K_e as for the $H^1(\Omega)$, the full mass matrix is obtained by summing up the contributions from the different elements.

The procedure is the same for the A_1 matrix. Let us here simply compute the expression of the element matrix, using the expression of the curl given in (1.40):

$$\begin{aligned}
\int_{K_e} \nabla\times\boldsymbol{\varphi}_i^1 \cdot \nabla\times\boldsymbol{\varphi}_j^1 \,\mathrm{d}\mathbf{x} &= \int_{\hat{K}} \frac{1}{J_e}(D\mathbf{F}_e)\hat{\nabla}\times\hat{\boldsymbol{\varphi}}_i^1 \cdot \frac{1}{J_e}(D\mathbf{F}_e)\hat{\nabla}\times\hat{\boldsymbol{\varphi}}_j^1 J_e \,\mathrm{d}\hat{\mathbf{x}},\\
&= \int_{\hat{K}} (c_{i,1}\mathbf{t}_1 + c_{i,2}\mathbf{t}_2 + c_{i,3}\mathbf{t}_3)\cdot(c_{j,1}\mathbf{t}_1 + c_{j,2}\mathbf{t}_2 + c_{j,3}\mathbf{t}_3)\frac{1}{J_e}\,\mathrm{d}\hat{\mathbf{x}},\\
&= \int_{\hat{K}} (\hat{\nabla}\times\hat{\boldsymbol{\varphi}}_i^1)^\top G_t^e \hat{\nabla}\times\hat{\boldsymbol{\varphi}}_j^1 \frac{1}{J_e}\,\mathrm{d}\hat{\mathbf{x}},
\end{aligned}$$

where we have denoted $\hat{\nabla}\times\hat{\boldsymbol{\varphi}}_i = (c_{i,1}, c_{i,2}, c_{i,3})^\top$ and $G_t^e = ((\mathbf{t}_i\cdot\mathbf{t}_j))_{1\leq i,j\leq 3}$ is the covariant metric tensor.

1.7.5 $H(\mathbf{div},\Omega)$ conforming Finite Elements

The $H(\mathrm{div},\Omega)$ function space and its conforming approximation also consist of vectors. Hence on a change of basis, one needs to decide how the components of the vectors are transformed. In order to have a natural map of the normal to a face used in the definition of the degrees of freedom and that guarantee conformity, the vectors defining the edges of the reference element needed to be mapped to

either the covariant or the contravariant basis associated to the mapping. In the $H(\mathrm{div}, \Omega)$ case it turns out that the right basis is the covariant basis $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$. From the expression of the transformation rule for the curl (1.40) we see that components are exactly conserved if in addition we scale the length by $1/J(\hat{\mathbf{x}})$. This then leads to the following transformation rule

$$\mathbf{v}(\mathbf{x}) = \frac{1}{J(\hat{\mathbf{x}})} D\mathbf{F}(\hat{\mathbf{x}})\hat{\mathbf{v}}(\hat{\mathbf{x}}), \qquad \text{with } \mathbf{x} = F(\hat{\mathbf{x}}). \tag{1.45}$$

This transformation is called the contravariant Piola transform. In the context of differential forms it is the natural pullback transformation of a 2-form (in a 3D space).

In order to get an expression of a variational formulation defined in $H(\mathrm{div}, \Omega)$, we now need to compute the curl of a vector in this basis.

Proposition 4 *We have the following properties*

(i) The jacobian matrix of a vector field transforms in the following way

$$D\mathbf{v} = T + \frac{1}{J(\hat{\mathbf{x}})} (D\mathbf{F}(\hat{\mathbf{x}}))^{-1} \hat{D}\hat{\mathbf{v}}(\hat{\mathbf{x}})(D\mathbf{F}(\hat{\mathbf{x}})), \tag{1.46}$$

where T is a matrix with vanishing trace.

(ii) The divergence transforms under the transformation rule (1.45)

$$\nabla \cdot \mathbf{v}(\hat{\mathbf{x}}) = \frac{1}{J(\hat{\mathbf{x}})} \hat{\nabla} \cdot \hat{\mathbf{v}}(\hat{\mathbf{x}}). \tag{1.47}$$

Proof. (i) The vector field $\mathbf{v}$ given by transform (1.45) can also be expressed on the covariant basis by

$$\mathbf{v} = \frac{1}{J}(\hat{v}_1 \mathbf{t}_1 + \hat{v}_2 \mathbf{t}_2 + \hat{v}_3 \mathbf{t}_3).$$

Taking the dot product with $\mathbf{n}_i$ yields $\hat{v}_i = J\mathbf{n}_i \cdot \mathbf{v}$, so that the derivative becomes

$$\frac{\partial \hat{v}_i}{\partial \hat{x}_j} = \frac{\partial J\mathbf{n}_i}{\partial \hat{x}_j} \cdot \mathbf{v} + J\mathbf{n}_i \cdot \frac{\partial \mathbf{v}}{\partial \hat{x}_j} = \frac{\partial J\mathbf{n}_i}{\partial \hat{x}_j} \cdot \mathbf{v} + J \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial \hat{x}_i}{\partial x_k} \frac{\partial v_k}{\partial x_l} \frac{\partial x_l}{\partial \hat{x}_j},$$

as $\mathbf{n}_i = \left(\frac{\partial \hat{x}_i}{\partial x_1}, \frac{\partial \hat{x}_i}{\partial x_2}, \frac{\partial \hat{x}_i}{\partial x_3} \right)^\top$. Now denoting by $\tilde{T} = ((\frac{\partial J\mathbf{n}_i}{\partial \hat{x}_j} \cdot \mathbf{v}))_{1 \le i,j \le 3}$, this implies that

$$\hat{D}\hat{\mathbf{v}} = \tilde{T} + J(D\mathbf{F}) D\mathbf{v} (D\mathbf{F})^{-1}.$$

Let us not verify that the trace of $\tilde{T}$ vanishes. To this aim let us first recall that

$$J\mathbf{n}_1 = \mathbf{t}_2 \times \mathbf{t}_3, \quad J\mathbf{n}_2 = \mathbf{t}_3 \times \mathbf{t}_1, \quad J\mathbf{n}_3 = \mathbf{t}_1 \times \mathbf{t}_2.$$

Moreover from the expression of $\mathbf{t}_j$ we find that $\frac{\partial \mathbf{t}_i}{\partial \hat{x}_j} = \frac{\partial \mathbf{t}_j}{\partial \hat{x}_i}$, so that

$$\begin{aligned}
\operatorname{tr} \tilde{T} &= \left(\frac{\partial \mathbf{t}_2}{\partial \hat{x}_1} \times \mathbf{t}_3 + \mathbf{t}_2 \times \frac{\partial \mathbf{t}_3}{\partial \hat{x}_1} + \frac{\partial \mathbf{t}_3}{\partial \hat{x}_2} \times \mathbf{t}_1 + \mathbf{t}_3 \times \frac{\partial \mathbf{t}_1}{\partial \hat{x}_2} + \frac{\partial \mathbf{t}_1}{\partial \hat{x}_3} \times \mathbf{t}_2 + \mathbf{t}_1 \times \frac{\partial \mathbf{t}_2}{\partial \hat{x}_3} \right) \cdot \mathbf{v}, \\
&= \left(\frac{\partial \mathbf{t}_2}{\partial \hat{x}_1} \times \mathbf{t}_3 + \mathbf{t}_3 \times \frac{\partial \mathbf{t}_2}{\partial \hat{x}_1} + \mathbf{t}_2 \times \frac{\partial \mathbf{t}_3}{\partial \hat{x}_1} + \frac{\partial \mathbf{t}_3}{\partial \hat{x}_1} \times \mathbf{t}_2 + \frac{\partial \mathbf{t}_3}{\partial \hat{x}_2} \times \mathbf{t}_1 + \mathbf{t}_1 \times \frac{\partial \mathbf{t}_3}{\partial \hat{x}_2} \right) \cdot \mathbf{v}, \\
&= 0.
\end{aligned}$$

We get the desired result by dividing by j, multiplying by $(D\mathbf{F})^{-1}$ on the left and by $(D\mathbf{F})$. Indeed the matrix T obtained from $\tilde{T}$ in this process is similar to $\tilde{T}$ up to the scalar scaling factor J and as similar matrices have the same trace it follows that $\operatorname{tr} T = 0$.

(ii) Relation (1.47) is obtained by taking the trace of (1.46) and using again that similar matrices have the same trace and that $\operatorname{tr} T = 0$

■

Now using transformation (1.45) one can check by direct computation that all terms appearing in the divergence Green's formula (1.7) including the normal component are exactly preserved.

Proposition 5 *We have the following properties:*

(i)

$$\int_K \mathbf{u} \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} = \int_{\hat{K}} \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\varphi} \, \mathrm{d}\hat{\mathbf{x}} \tag{1.48}$$

(ii)

$$\int_K \nabla \cdot \mathbf{u} \, \varphi \, \mathrm{d}\mathbf{x} = \int_{\hat{K}} \hat{\nabla} \cdot \hat{\mathbf{u}} \, \hat{\varphi} \, \mathrm{d}\hat{\mathbf{x}} \tag{1.49}$$

(iii) For any face f_i on the boundary of K image of the face $\hat{f}_i$ of the reference element

$$\int_{f_i} \mathbf{u} \cdot \boldsymbol{\nu} \, \varphi \, \mathrm{d}\boldsymbol{\sigma} = \int_{\hat{f}_i} \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\nu}} \, \hat{\varphi} \, \mathrm{d}\hat{\boldsymbol{\sigma}} \tag{1.50}$$

We observe in particular, that all these terms are completely independent of the metric and have the same expression under any mapping.

Proof. (i) Let us as usual perform the change of variables $\mathbf{x} = \mathbf{F}(\hat{\mathbf{x}})$ $\mathrm{d}\mathbf{x} = J(\hat{\mathbf{x}}) \, \mathrm{d}\hat{\mathbf{x}}$, using the transformation rule (1.45) for $\mathbf{u}$ in the integral

$$\int_K \mathbf{u} \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} = \int_{\hat{K}} \frac{1}{J} (D\mathbf{F}) \hat{\mathbf{u}} \cdot (D\mathbf{F})^{-\top} \hat{\nabla} \hat{\varphi} J \, \mathrm{d}\hat{\mathbf{x}} = \int_{\hat{K}} \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\varphi} \, \mathrm{d}\hat{\mathbf{x}}.$$

(ii) With the same change of variable and transformation rule for $\mathbf{u}$, using (1.47) we find

$$\int_K \nabla \cdot \mathbf{u} \, \varphi \, \mathrm{d}\mathbf{x} = \int_{\hat{K}} \frac{1}{J} \hat{\nabla} \cdot \hat{\mathbf{u}} \, \hat{\varphi} J \, \mathrm{d}\hat{\mathbf{x}} = \int_{\hat{K}} \hat{\nabla} \cdot \hat{\mathbf{u}} \, \hat{\varphi} \, \mathrm{d}\hat{\mathbf{x}}$$

(iii) Here the face f_i is defined as a parametrized surface, the parametrization being given by one of the faces of the reference element. Let us do the computation for the face on the $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ plane for with the normal vector is $\hat{\boldsymbol{\nu}} = -\hat{\mathbf{e}}_3$. The parametrization is then defined by

$$(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) \mapsto \mathbf{F}(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, 0)$$

and the measure is

$$\mathrm{d}\boldsymbol{\sigma} = \|\mathbf{t}_1 \times \mathbf{t}_2\| \,\mathrm{d}\hat{\mathbf{x}}_1 \,\mathrm{d}\hat{\mathbf{x}}_2 = J\|\mathbf{n}_3\| \,\mathrm{d}\hat{\mathbf{x}}_1 \,\mathrm{d}\hat{\mathbf{x}}_2.$$

Moreover the outward unit normal vector is related to the contravariant vector $\mathbf{n}_3$, which is by definition orthogonal to the face, by $\nu = -\mathbf{n}_3/\|\mathbf{n}_3\|$. Finally, applying the transformation rule (1.45) for $\mathbf{u}$ we get

$$\begin{aligned}
&\int_{f_i} \mathbf{u} \cdot \boldsymbol{\nu}\, \varphi \,\mathrm{d}\boldsymbol{\sigma} = -\int_{\hat{f}_i} \frac{1}{J}(D\mathbf{F})\hat{\mathbf{u}} \cdot \frac{\mathbf{n}_3}{\|\mathbf{n}_3\|} \hat{\varphi} J \|\mathbf{n}_3\| \,\mathrm{d}\hat{\mathbf{x}}_1 \,\mathrm{d}\hat{\mathbf{x}}_2 \\
&= -\int_{\hat{f}_i} (\hat{u}_1\mathbf{t}_1 + \hat{u}_2\mathbf{t}_2 + \hat{u}_3\mathbf{t}_3)\cdot\mathbf{n}_3\hat{\varphi} \,\mathrm{d}\hat{\mathbf{x}}_1 \,\mathrm{d}\hat{\mathbf{x}}_2 = -\int_{\hat{f}_i} \hat{u}_3\hat{\varphi} \,\mathrm{d}\hat{\mathbf{x}}_1 \,\mathrm{d}\hat{\mathbf{x}}_2 = \int_{\hat{f}_i} \hat{\mathbf{u}}\cdot\hat{\boldsymbol{\nu}}\, \hat{\varphi} \,\mathrm{d}\hat{\mathbf{x}}_1 \,\mathrm{d}\hat{\mathbf{x}}_2
\end{aligned}$$

■

To conclude, we assemble the matrices involved in a variational formulation posed on $H(\mathrm{div}, \Omega)$. Let us consider for example the variational formulation
Find $\mathbf{u} \in H(div, \Omega)$ *such that*

$$\int_\Omega \mathbf{u} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} + \int_\Omega \nabla \cdot \mathbf{u} \cdot \nabla \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in H(\mathrm{div}, \Omega). \tag{1.51}$$

We need to build the conforming Finite Element space $V_h^2 \subset H(\mathrm{div}, \Omega)$ based on the $H(\mathrm{div}, \Omega)$ reference element $(\hat{K}, \hat{P}, \hat{\Sigma})$. For this we consider as previously a mesh denoted by $\mathcal{T} = \bigcup_{1\le e\le N_{el}} K_i$ consisting of N_{el} disjoint conforming elements denoted each by K_i. On this mesh we can then define $V_h^2 \subset H(\mathrm{div}, \Omega)$ by

$$V_h^2 = \{\mathbf{v}_h \in H(\mathrm{div}, \Omega) \,|\, \mathbf{v}_{h|K_e} = \frac{1}{J}(D\mathbf{F})\hat{\mathbf{v}}_e \circ F_e^{-1},\ \hat{\mathbf{v}}_e \in \hat{P},\quad 1 \le e \le N_{el}\}.$$

Now we can in the variational formulation (1.44) replace $H(\mathrm{div}, \Omega)$ by V_h^2 and $\mathbf{u}$ and $\mathbf{v}$ by their discrete counterparts

$$\mathbf{u}_h(\mathbf{x}) = \sum_{j=1}^{N_2} u_j \boldsymbol{\varphi}_j^2(\mathbf{x}), \qquad \mathbf{v}_h(\mathbf{x}) = \sum_{i=1}^{N_2} v_i \boldsymbol{\varphi}_i^2(\mathbf{x}),$$

where $(\boldsymbol{\varphi}_i^2)_{1\le i\le N_2}$ stands for the basis of V_h^2 associated to the Finite Element. As for the $H^1(\Omega)$ and $H(\mathrm{curl}, \Omega)$ conforming case, plugging these expressions into the variational formulation leads to the matrix expression:

$$V^\top M_2 U + V^\top A_2 U = V^\top \mathbf{b},$$

where $M_2 = ((\int_\Omega \boldsymbol{\varphi}_i^2 \cdot \boldsymbol{\varphi}_j^2 \,\mathrm{d}\mathbf{x}))_{i,j}$ is the mass matrix in V_h^2, $A_2 = ((\int_\Omega \nabla \cdot \boldsymbol{\varphi}_i^2 \nabla \cdot \boldsymbol{\varphi}_j^2 \,\mathrm{d}\mathbf{x}))_{i,j}$, and $b = (\int \mathbf{f} \cdot \boldsymbol{\varphi}_i^2)_i$. There now only remains to compute these integrals. Let us start with the mass matrix M_2

$$\int_\Omega \boldsymbol{\varphi}_i^2 \cdot \boldsymbol{\varphi}_j^2 \,\mathrm{d}\mathbf{x} = \sum_{e=1}^{N_{el}} \int_{K_e} \boldsymbol{\varphi}_i^2 \cdot \boldsymbol{\varphi}_j^2 \,\mathrm{d}\mathbf{x}.$$

Then on each element K_e we make the change of variables $\mathbf{x} = \mathbf{F}_e(\hat{\mathbf{x}})$, $\mathrm{d}\mathbf{x} = J_e(\hat{\mathbf{x}}) \,\mathrm{d}\hat{\mathbf{x}}$, using the contravariant Piola transform (1.45) for the vectors to get

$$\begin{aligned}
\int_{K_e} \boldsymbol{\varphi}_i^2 \cdot \boldsymbol{\varphi}_j^2 \,\mathrm{d}\mathbf{x} &= \int_{\hat{K}} \frac{1}{J_e}(D\mathbf{F}_e)\hat{\boldsymbol{\varphi}}_i^2 \cdot \frac{1}{J_e}(D\mathbf{F}_e)\hat{\boldsymbol{\varphi}}_j^2 J_e \,\mathrm{d}\hat{\mathbf{x}}, \\
&= \int_{\hat{K}} \frac{1}{J_e}(\hat{\varphi}_{i,1}^2 \mathbf{t}_1 + \hat{\varphi}_{i,2}^2 \mathbf{t}_2 + \hat{\varphi}_{i,3}^2 \mathbf{t}_3) \cdot (\hat{\varphi}_{j,1}^2 \mathbf{t}_1 + \hat{\varphi}_{j,2}^2 \mathbf{t}_2 + \hat{\varphi}_{j,3}^2 \mathbf{t}_3) \,\mathrm{d}\hat{\mathbf{x}}, \\
&= \int_{\hat{K}} (\hat{\boldsymbol{\varphi}}_i^2)^\top G_t \hat{\boldsymbol{\varphi}}_j^2 \frac{1}{J_e} \,\mathrm{d}\hat{\mathbf{x}},
\end{aligned}$$

where $G_t = ((\mathbf{t}_i \cdot \mathbf{t}_j))_{1 \le i,j \le 3j}$ is the covariant metric tensor. This is the element mass matrix on the element K_e as for the $H^1(\Omega)$, the full mass matrix is obtained by summing up the contributions from the different elements.

The procedure is the same for the A_2 matrix. Let us here compute the expression of the element matrix, using the expression of the divergence given in (1.47):

$$\begin{aligned}
\int_{K_e} \nabla \cdot \boldsymbol{\varphi}_i^2 \nabla \cdot \boldsymbol{\varphi}_j^2 \,\mathrm{d}\mathbf{x} &= \int_{\hat{K}} \frac{1}{J_e} \hat{\nabla} \cdot \hat{\boldsymbol{\varphi}}_i^2 \frac{1}{J_e} \hat{\nabla} \cdot \hat{\boldsymbol{\varphi}}_j^2 J_e \,\mathrm{d}\hat{\mathbf{x}}, \\
&= \int_{\hat{K}} \frac{1}{J_e} \hat{\nabla} \cdot \hat{\boldsymbol{\varphi}}_i^2 \,\hat{\nabla} \cdot \hat{\boldsymbol{\varphi}}_j^2 \,\mathrm{d}\hat{\mathbf{x}}.
\end{aligned}$$

Note that here the divergence is a scalar. There is no metric tensor in the matrix. Only the inverse Jacobian is present because of the mapping.

1.8 Convergence of the Finite Element method

The Ritz-Galerkin method consists in finding an approximate solution u_h in a finite dimensional subspace of V. For convergence studies one needs to consider a sequence of subspaces of V of larger and larger dimension so that they get closer to V. One then defines a sequence of problems parametrised by h that read:
Find $u_h \in V_h$ such that

$$a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h, \tag{1.52}$$

where $V_h \subset V$ is a vector space of dimension N. Let $(\varphi_1, \ldots, \varphi_N)$ a basis of V_h. An element $u_h \in V_h$ can then be expanded as $u_h(x) = \sum_{j=1}^N u_j \varphi_j(x)$. Taking $v_h = \varphi_i$ the equation (1.52) becomes using the linearity

$$\sum_{j=1}^N u_j a(\varphi_j, \varphi_i) = l(\varphi_i).$$

Then using the symmetry of a, we notice that the discrete variational formulation(1.52) is equivalent to the linear system

$$AU_h = L, \tag{1.53}$$

where $A = (a(\varphi_i, \varphi_j))_{1 \leq i,j \leq N}$, L is the column vector with components $l(\varphi_i)$ and U is the column vector with the unknowns u_i that are the coefficients of u_h in the basis $(\varphi_1, \ldots, \varphi_N)$.

Theorem 2 *Assume that a is a symmetric continuous and coercive bilinear form on a Hilbert space V and l a continuous linear form on V. Then the system (1.53) is equivalent to the discrete variational form (1.52) and admits a unique solution*

Proof. For $v_h \in V_h$, we denote by $\tilde{V}$ the vector of its components in the basis $(\varphi_1, \ldots, \varphi_N)$.

- Thanks to the bilinearity of a and the linearity of l the relation (1.52) can be written equivalently

  $$^t\tilde{V}AU_h = {}^t\tilde{V}L \quad \forall \tilde{V} \in \mathbb{R}^N, \tag{1.54}$$

  which means that the vector $AU_h - L \in \mathbb{R}^N$ is orthogonal to all the vectors of $\mathbb{R}^N$, and so is the zero vector. Conversely it is clear that (1.53) implies (1.54) and so (1.52).

- Let $v_h \in V_h$. Then, as a is coercive, there exists $\alpha > 0$ such that

  $$^t\tilde{V}A\tilde{V} = a(v_h, v_h) \geq \alpha \|v_h\|^2 \geq 0,$$

  and $^t\tilde{V}A\tilde{V} = 0 = a(v_h, v_h) \Rightarrow \|v_h\| = 0$, which implies that $v_h = 0$ and so $\tilde{V} = 0$. So A is symmetric, positive definite and therefore invertible.

■

After making sure the approximate solution exists for some given space V_h, one needs to make sure the approximation converges towards the exact solution. This results from two properties: 1) The Galerkin orthogonality, which comes from the conforming Gakerkin approximation, 2) The approximability property, which makes sure that for any $v \in V$ there exist v_h in some finite dimensional space of the family which is close enough to v.

Lemma 9 (Céa) *Let $u \in V$ the solution of (1.19) and $u_h \in V_h$ the solution of (1.52), with $V_h \subset V$. Then*

$$\|u - u_h\| \leq C \inf_{v_h \in V_h} \|u - v_h\|.$$

Proof. We have

$$a(u,v) = l(v) \quad \forall v \in V,$$
$$a(u_h,v_h) = l(v_h) \quad \forall v_h \in V_h,$$

as $V_h \subset V$, we can take $v = v_h$ in the first equality and take the difference which yields

$$a(u-u_h,v_h) = 0 \quad \forall v_h \in V_h.$$

It results that $a(u-u_h,u-u_h) = a(u-u_h,u-v_h+v_h-u_h) = a(u-u_h,u-v_h)$, as $v_h - u_h \in V_h$ and so $a(u-u_h,v_h-u_h) = 0$. Then there exists $\alpha > 0$ and β such that

$$\begin{aligned}
\alpha\|u-u_h\|^2 &\leq a(u-u_h,u-u_h) && \text{as } a \text{ is coercive,}\\
&\leq a(u-u_h,u-v_h) \quad \forall v_h \in V_h,\\
&\leq \beta\|u-u_h\|\|u-v_h\| && \text{as } a \text{ is continuous.}
\end{aligned}$$

Whence $\|u-u_h\| \leq \frac{\beta}{\alpha}\|u-v_h\|$ for all $v_h \in V_h$. We get the desired results taking the infimum in V_h. ∎

For the global error estimates, we make the following hypotheses on the triangulation $\mathcal{T}_h$, where we denote by h_K the diameter of the circumscribed circle and ρ_K the diameter of the inscribed circle of triangle K:

(H1) We assume that the family of triangulations is regular in the following sense:

(i) There existes a constant σ such that

$$\forall K \in \cup_h \mathcal{T}_h \quad \frac{h_K}{\rho_K} \leq \sigma.$$

(ii) The quantity $h = \max_{K\in_h} h_K$ tend to 0.

(H2) All finite elements (K,P,Σ), $K \in \cup_h \mathcal{T}_h$ are affine equivalent to a unique reference element $(\hat{K},\hat{P},\hat{\Sigma})$.

(H3) All finite elements (K,P,Σ), $K \in \cup_h \mathcal{T}_h$ are of class C^0.

Theorem 3 *We assume the hypotheses (H1), (H2) and (H3) are verified. Moreover we assume that there exists an integer $k \geq 1$ such that*

$$\mathbb{P}_k \subset \hat{P} \subset H^1(\hat{K}),$$

$$H^{k+1}(\hat{K}) \subset C^0(\hat{K}) \qquad (\textit{true if } k+1 > \frac{d}{2}, \hat{K} \subset \mathbb{R}^d).$$

Then there exists a constant C independent of h such that for any function $v \in H^{k+1}(\Omega)$ we have

$$\|v-\pi_h v\|_0 \leq Ch^{k+1}|v|_{k+1,\Omega}, \qquad |v-\pi_h v|_1 \leq Ch^k|v|_{k+1,\Omega},$$

where π_h is the finite element interpolation operator of the finite element (K, P, Σ) defined by

$$\pi_h v = \sum_{i=1}^{N} \sigma_i(v)\, p_i,$$

where $\sigma_i \in \Sigma$ and p_i the corresponding basis function of P.

We consider a variational problem posed in $V \subset H^1(\Omega)$.

Theorem 4 *We assume that (H1), (H2) and (H3) are verified. Moreover we assume that there exists an integer $k \geq 1$ such that $k + 1 > \frac{d}{2}$ with $\mathbb{P}_k(\hat{K}) \subset P \subset H^1(\hat{K})$ and that the exact solution of the variational problem is in $H^{k+1}(\Omega)$, then*

$$\|u - u_h\|_{1,\Omega} \leq C h^k |u|_{k+1,\Omega},$$

where $u_h \in V_h$ is the discrete solution.

Proof. We have because of the polynomial approximation theorem

$$\|u - \pi_h u\|_{1,\Omega} \leq C h^k |u|_{k+1,\Omega}.$$

On the other hand Céa's lemma gives us

$$\|u - u_h\|_{1,\Omega} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} \leq C \|u - \pi_h u\|_{1,\Omega}.$$

The result follows. ∎

1.9 The discontinuous Galerkin (DG) method

In some cases it can be convenient to relax the constraints of the approximation space being a subspace of V or to slightly modify the bilinear and linear form, using for example quadrature rules to compute the integrals or adding penalty terms. Let us then consider a mesh dependent family of Hilbert spaces H_h equipped with a mesh dependent norm $\|\cdot\|_h$ such that $V \subset H_h$ and also $V_h \subset H_h$ for all h. One could choose $H_h = V + V_h$, but often some additional regularity of the exact solution is needed. Then one can choose $H_h = V \cap H^m(\Omega) + V_h$, where H^m is a Sobolev space of the needed regularity. We then define the bilinear form a_h on H_h as an approximation of the bilinear form a on V, and l_h a linear form on H_h approximating l the linear form on V. We assume that a_h and l_h are continuous and that a_h is uniformly coercive on H_h, meaning that the coercivity constant does not depend on h, even though the norm itself can depend on h.

For discontinuous Finite Elements approximation a variational formulation in H^1, given a subdivision of the domain Ω into elements $\Omega = \cup K_e$ we can define

$$H_h = \{v \in L^2(\Omega) \mid v_{|K_e} \in H^1(K_e)\}, \qquad \text{with } \|v\|_h^2 = \sum_e \int_{K_e} |\nabla v(x)|^2 \,\mathrm{d}x + \int_\Omega v^2 \,\mathrm{d}x.$$

This is a broken H^1 space. Its elements are locally H^1 but not globally. Obviously $V = H^1(\Omega) \subset H_h$. Depending on our problem we can then define a mesh dependent bilinear form a_h and a mesh dependent linear form l_h. In this setting, approximating the variational formulation *Find $u \in V$ such that*

$$a(u, v) = l(v) \quad \forall v \in V, \tag{1.55}$$

we consider a discrete variational formulation of the following form:
Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h. \tag{1.56}$$

The error estimation of such an approximate variational formulation depends on the following lemma, known as Strang's second lemma [14], which replaces Céa's lemma:

Lemma 10 (Strang) *Assume $u \in V$ is the solution of* (1.55) *and $u_h \in V_h$ its approximation, solution of* (1.56)*, then*

$$\|u - u_h\|_h \leq C \left(\inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{v_h \in V_h} \frac{|a_h(u, v_h) - l_h(v_h)|}{\|v_h\|_h} \right). \tag{1.57}$$

Note that as u_h verifies $a_h(u_h, v_h) = l_h(v_h)$, we also could write $a_h(u, v_h) - l_h(v_h) = a_h(u - u_h, v_h)$.

Remark 8 *The difference between Céa's Lemma that is used for the conforming approximation and the above Strang Lemma used for non conforming approximation is the presence of the second term on the right-hand-side, which is a consistency term, measuring how accurately the continuous solution verifies the discrete equation. This term vanishes when $V_h \subset V$ and $a_h = a$ because of the Galerkin orthogonality.*

Proof. By bilinearity of a_h, we have

$$a_h(v_h - u_h, v_h - u_h) = a_h(v_h - u + u - u_h, v_h - u_h) = a_h(v_h - u, v_h - u_h) + a_h(u - u_h, v_h - u_h)$$

and also

$$\begin{aligned} a_h(u - u_h, v_h - u_h) &= a_h(u, v_h - u_h) - a_h(u_h, v_h - u_h), \\ &= a_h(u, v_h - u_h) - l_h(v_h - u_h), \end{aligned}$$

as u_h verifies (1.56).

Using the uniform coercivity and continuity of a_h in H_h, there exists $\alpha > 0$ and β such that for any $v_h \in V_h$

$$\begin{aligned} \alpha \|v_h - u_h\|_h^2 &\leq a_h(v_h - u_h, v_h - u_h) \leq a_h(v_h - u, v_h - u_h) + a_h(u - u_h, v_h - u_h) \\ &\leq a_h(v_h - u, v_h - u_h) + a_h(u, v_h - u_h) - l_h(v_h - u_h), \\ &\leq \beta \|v_h - u\|_h \|v_h - u_h\|_h + \frac{|a_h(u, v_h - u_h) - l_h(v_h - u_h)|}{\|v_h - u_h\|_h} \|v_h - u_h\|_h \end{aligned}$$

as a_h is continuous. Whence

$$\|v_h - u_h\|_h \leq \frac{\beta}{\alpha}\|u - v_h\|_h + \frac{1}{\alpha}\sup_{v_h \in V_h} \frac{|a_h(u, v_h) - l_h(v_h)|}{\|v_h\|_h}$$

for all $v_h \in V_h$. This combined with the triangle inequality yields

$$\|u-u_h\|_h \leq \|u-v_h\|_h + \|v_h-u_h\|_h \leq \left(1 + \frac{\beta}{\alpha}\right)\|u-v_h\|_h + \frac{1}{\alpha}\sup_{v_h \in V_h} \frac{|a_h(u, v_h) - l_h(v_h)|}{\|v_h\|_h}.$$

We then get the desired results taking the infimum in V_h. ∎

Now using our broken space H_h, we want to define a mesh dependent bilinear form a_h on H_h which is uniformly coercive (*i.e.* with a coercivity constant that does not depend on the mesh) and such that the consistency error, which is the second term on the right-hand-side of (1.57), vanishes. To achieve consistency it is convenient to derive the bilinear form starting from the exact solution. Then we can symmetrize and penalize the obtained formulation to get a symmetric and coercive mesh dependent form. This leads to the so-called SIP (symmetric interior penalty) Discontinuous Galerkin (DG) method that was introduced, adapting previous works, and analyzed by Arnold [2]. See also [3] for a historical and more general discussion on Discontinuous Galerkin methods. A mathematical textbook presentation of the discontinuous Galerkin method can be found for example in [9].

Let us explain the method on the Poisson equation with homogeneous Dirichlet boundary conditions.

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega.$$

Instead of multiplying by a smooth function over the whole domain as in the classical finite element method, we here introduce the mesh first $\mathcal{T}_h = \cup_e K_e$. Let $v_h \in L^2(\Omega)$ such that $v_h \in H^1(K_e)$. In practice v_h will be polynomial or in a local Finite Element space. Let us denote by n_e the outward unit normal for element K_e and denoting by $(\Gamma_s)_s$ the edges of the grid, $\Gamma_s = K_{e_1} \cap K_{e_2}$, by n_s a unit normal to the edge Γ_s (the sign being arbitrarily chosen so that $n_s = n_{e_1}$). As usual in Discontinuous Galerkin formulations we denote by $\{\{v\}\} = (v_{|K_{e_1}} + v_{|K_{e_2}})/2$ the average of v on the edge Γ_s and $[\![v]\!] = v_{|K_{e_1}} - v_{|K_{e_2}}$ the jump of v on the edge Γ_s. For edges on the boundary of the domain, we assume that the outside value is zero so that $\{\{v\}\} = [\![v]\!] = v$. Then, we multiply by a smooth function $v_{h|K_e}$ on one element, integrate over this element and sum over all the elements

$$\begin{aligned}
\sum_e \int_{K_e} (-\Delta u) v_h \,\mathrm{d}\mathbf{x} &= \sum_e \left[\int_{K_e} \nabla u \cdot \nabla v_h \,\mathrm{d}\mathbf{x} - \int_{\partial K_e} \frac{\partial u}{\partial n_e} v_h \,\mathrm{d}\sigma\right] \\
&= \sum_e \int_{K_e} \nabla u \cdot \nabla v_h \,\mathrm{d}\mathbf{x} - \sum_s \int_{\Gamma_s} \frac{\partial u}{\partial n_s} [\![v_h]\!] \,\mathrm{d}\sigma \\
&= \sum_e \int_{K_e} \nabla u \cdot \nabla v_h \,\mathrm{d}\mathbf{x} - \sum_s \int_{\Gamma_s} \left\{\!\!\left\{ \frac{\partial u}{\partial n_s} \right\}\!\!\right\} [\![v_h]\!] \,\mathrm{d}\sigma \qquad (1.58)
\end{aligned}$$

as the exact solution, which is in H^2, has a continuous normal derivative across the edges, this is equal to its average.

This formula can be used to get a consistent approximation. In order to get coercivity, we symmetrize and penalize the interface jumps. This doesn't break consistency, as the jumps vanish for the exact solution. We then get the mesh dependent bilinear form

$$a_h(u_h, v_h) = \sum_e \int_{K_e} (\nabla u \cdot \nabla v_h)\,\mathrm{d}\mathbf{x} - \sum_s \int_{\Gamma_s} \left\{\!\!\left\{ \frac{\partial u_h}{\partial n_s} \right\}\!\!\right\} [\![v_h]\!]\,\mathrm{d}\sigma - \sum_s \int_{\Gamma_s} [\![u_h]\!] \left\{\!\!\left\{ \frac{\partial v_h}{\partial n_s} \right\}\!\!\right\} \mathrm{d}\sigma + \mu_h \int_{\Gamma_s} [\![u_h]\!]\,[\![v_h]\!]\,\mathrm{d}\sigma. \quad (1.59)$$

This is obviously symmetric, and can be proven to be uniformly coercive provided the mesh dependent penalty term μ_h is appropriately chosen.

Then, defining the linear form

$$l_h(v_h) = \int_\Omega f v_h\,\mathrm{d}\mathbf{x} = \sum_e \int_{K_e} f v_h\,\mathrm{d}\mathbf{x}$$

and the discontinuous polynomial space

$$V_h = \{v_h \in L^2(\Omega) \mid v_{h|K_e} \in \mathbb{P}_k(K_e)\},$$

the discrete variational formulation for our problem reads: *Find* $u_h \in V_h$ *such that*

$$a_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h. \quad (1.60)$$

As the exact solution u is continuous, the jump of u at the element interfaces vanishes, *i.e.* $[\![u]\!] = 0$, hence the two last terms in (1.59), which are respectively the symmetry term and the penalty term vanish for $u_h = u$. Hence using the calculation (1.58) we can verify that the consistency error in the DG-SIP method (1.60) is exactly vanishing:

$$a_h(u, v_h) - l_h(v_h) = \sum_e \int_{K_e} (-\Delta u) v_h\,\mathrm{d}\mathbf{x} - \int_\Omega f v_h\,\mathrm{d}\mathbf{x} = -\int_\Omega (\Delta u + f) v_h\,\mathrm{d}\mathbf{x} = 0$$

as $-\Delta u = f$.

Even though they are not conforming as $V_h \not\subset V$, DG methods are build to be exactly consistent, so that the consistency error in the second Strang lemma 10 exactly vanishes. The error analysis described in the book by Di Pietro and Ern [9] relies on this property. See in particular Section 4.2, which provides a convergence analysis of the SIP-DG method in an *ad hoc* mesh dependent norm, following the same pattern as for the Nitsche method.

Chapter 2

Finite Element Exterior Calculus (FEEC)

The aim of this chapter is to provide a different point of view to the Finite Element spaces, in particular to the vector based Finite Element spaces $H(\mathrm{curl}, \Omega)$ and $H(\mathrm{div}, \Omega)$. We will also see that the differential forms interpretation naturally leads to a sequence of spaces called differential complex and related with a differential operator.

We will consider here mostly the Isogeometric Analysis point of view, where we have a physical mesh mapped from a cartesian logical mesh (see Figure 2.1). An important insight for Finite Elements is to understand what properties depend only on the connectivity of the mesh (topological properties invariant by continuous deformation of the mesh) and what properties depend on the mapping (geometrical properties).

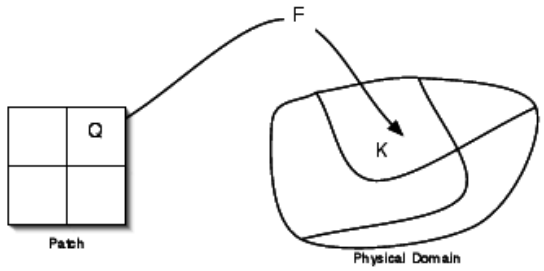

Figure 2.1: An example of a mapping from the logical to the physical mesh

2.1 Some tools from differential geometry

In this section we introduced the basic notions of differential geometry that will be needed for the Finite Element Exterior Calculus. We follow mostly the definitions from the book of Frankel [11] and the paper by Palha et al. [13] introducing so-called physics compatible discretization techniques.

The basic object of differential geometry is a manifold $\mathcal{M}$:

Definition 4 *A (differentiable) manifold of dimension n is defined as a collection of p mappings*

$$\begin{aligned}\mathbf{F}_\nu : U_\nu \subset \mathbb{R}^n &\to \mathcal{M},\\ \mathbf{q} &\mapsto \mathbf{F}_\nu(\mathbf{q}) = (x_1^\nu(\mathbf{q}), \ldots, x_n^\nu(\mathbf{q}))\end{aligned}$$

which are at least C^1 diffeomorphisms, that cover $\mathcal{M}$. The sets U_ν are open and the coordinate systems are compatible in the intersection of two such sets.

Each of the mapping $\mathbf{F}_\nu$ defines a local coordinate system (which is global if there is only one mapping). With the mapping come the Jacobians

$$D\mathbf{F}_\nu(\mathbf{q}) = ((\frac{\partial x_i^\nu}{\partial q_j}))_{1\le i,j\le n}, \qquad D(\mathbf{F}_\nu^{-1})(\mathbf{x}) = ((\frac{\partial q_i}{\partial x_j^\nu}))_{1\le i,j\le n}.$$

Using the chain rule on $\mathbf{F}_\nu^{-1}(\mathbf{F}_\nu(\mathbf{q})) = \mathbf{q}$ one finds that $D(\mathbf{F}_\nu^{-1})(\mathbf{x}) = (D\mathbf{F}_\nu(\mathbf{q}))^{-1}$.

In our application the manifold will just be $\mathbb{R}^n$ with a unique mapping $\mathbf{F}$ defining the coordinate system. Another useful example to understand the notion is the surface of a sphere, which is a two-dimensional manifold, that needs at least two mappings (also called charts in the differential geometry literature, where they are sometimes defined by the inverse of the mapping we are using here) to be completely covered without singular point. See Figure 2.2 for an illustration.

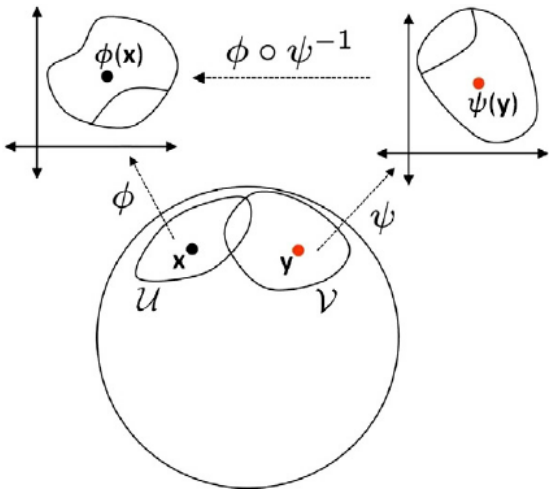

Figure 2.2: Illustration of two charts on a sphere.

An important goal in differential geometry is to define quantities independently of the local coordinates. Such quantities are called intrinsic.

Tangent plane and vectors. The first important object that can be associated to a point $\mathbf{x}$ in a manifold is a vector. A vector is intrinsically defined as the tangent at $\mathbf{x}$ to a curve passing through $\mathbf{x}$. In local coordinates a curve γ is parametrized by

$$\begin{aligned}\gamma :[a,b] &\to \mathcal{M}\\ s &\mapsto (x_1(\mathbf{q}(s)), \ldots x_n(\mathbf{q}(s)))\end{aligned}$$

and its tangent at $\mathbf{x} = \gamma(c)$, $c \in [a, b]$ is

$$\mathbf{v}(c) = \frac{\mathrm{d}\gamma}{\mathrm{d}s}(c) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}(c) = \left(\frac{\mathrm{d}x_1}{\mathrm{d}s}(c), \ldots, \frac{\mathrm{d}x_n}{\mathrm{d}s}(c) \right)^\top.$$

The collection of all possible curves passing through $\mathbf{x}$ on $\mathcal{M}$ define a vector for each curve and altogether the tangent plane $T_x\mathcal{M}$ at $\mathbf{x}$. If the $\mathcal{M} = \mathbb{R}^n$ as in our main application the tangent plane is also $T_x\mathcal{M} = \mathbb{R}^n$, however this is not always the case as one can figure out by considering the surface of a sphere as a manifold.

Considering not only the vector at a specific point $\mathbf{x} = \gamma(c)$ but all vectors defined by a curve $\mathbf{v}(s) = \frac{\mathrm{d}\gamma}{\mathrm{d}s}(s)$, $s \in [a, b]$ defines a *vector field*. The collection of all points associated to their tangent plane

$$T\mathcal{M} = \{(\mathbf{x}, \mathbf{v}), \quad \mathbf{x} \in \mathcal{M}, \mathbf{v} \in T_x\mathcal{M}\}$$

is called the *tangent bundle*.

The local coordinates defined by the mapping $\mathbf{F}$ also defines a basis of $T_x\mathcal{M}$. Indeed, using the mapping $\mathbf{x}(s) = \mathbf{F}(\mathbf{q}(s))$, we find

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \sum_{j=1}^{n} \frac{\mathrm{d}q^j}{\mathrm{d}s} \frac{\partial \mathbf{F}}{\partial q_j} = \sum_{j=1}^{n} v^j \frac{\partial \mathbf{F}}{\partial q_j}.$$

The corresponding basis vectors are $\mathbf{t}_j = \frac{\partial \mathbf{F}}{\partial q_j} := \boldsymbol{\partial}_j$. Geometrically the basis is associated to the curves obtained by letting all q_j but one constant. These basis vectors are denoted in the modern differential geometry literature by $\boldsymbol{\partial}_j$, that we will use in these notes, or by $\frac{\partial}{\partial q_j}$. So that a vector $\mathsf{v} \in T_x\mathcal{M}$ can be expressed in the local coordinate basis by

$$\mathsf{v} = \sum_{i=1}^{n} v^i \boldsymbol{\partial}_i = (D\mathbf{F})\mathbf{v}.$$

We denote here in serif font v, the geometrical object. One can think of it as the vector expressed in a reference frame, typically cartesian coordinates of $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$ for an object embedded in a higher dimension manifold (as the surface of a sphere). And we denote in bold font the components of a column vector in a local basis $\mathbf{v} = (v^1, \ldots, v^n)^\top$. The last form above is a matrix vector multiplication directly following from the definition of $\boldsymbol{\partial}_i$ as the i^{th} column of the Jacobian matrix $D\mathbf{F}$. As the geometric object is independent of the choice of local coordinates we have

$$\mathsf{v} = (D\mathbf{F})\mathbf{v} = (D\mathbf{F}')\mathbf{v}',$$

if $\mathbf{F}'$ defines another local coordinate system (mapping) in which $\mathbf{v}' = (v'^1, \ldots, v'^n)$ are the components of v. We follow here the convention of denoting the components of a tangent vector with upper indices. Vectors or tangent vectors are also called contravariant vectors in the physics literature.

Example: Consider the mapping yielding polar coordinates

$$\mathbf{F}(r,\theta) = \begin{pmatrix} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{pmatrix}$$

A basis of the tangent plane at any point $\mathbf{x} = (x_1, x_2)$ is given by

$$\boldsymbol{\partial}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \boldsymbol{\partial}_\theta = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}.$$

Here $\boldsymbol{\partial}_r$ is tangent to the line $\theta = C$ (only r varies) passing through $\mathbf{x}$ and $\boldsymbol{\partial}_\theta$ is tangent to the line $r = C$ (only θ varies) passing through $\mathbf{x}$. Note that this basis is not normalized, as opposed to the classical basis used in polar coordinates $\mathbf{e}_r = (\cos\theta, \sin\theta)^\top$, $\mathbf{e}_\theta = (-\sin\theta, \cos\theta)^\top$. Moreover unlike in cartesian coordinates where the basis vectors of the tangent plane are constants, in polar coordinates, as in any general system of curvilinear coordinates, or local coordinates, they depend on the point of which the tangent plane is considered.

Cotangent plane and covectors. Even though vectors (tangent vectors) and vector fields are the most intuitive objects for defining dynamics on a manifold, an another object proved to be mathematically more interesting the *covector* p, which is defined as a linear form on the tangent space at $\mathbf{x}$, *i.e.* a covector applied to a vector yields a real number $\mathsf{p}(\mathsf{v}) \in \mathbb{R}$, and is linear: $\mathsf{p}(\lambda_1\mathsf{v}_1 + \lambda_2\mathsf{v}_2) = \lambda_1\mathsf{p}(\mathsf{v}_1) + \lambda_2\mathsf{p}(\mathsf{v}_2)$. The collection of all covectors at $\mathbf{x}$ is called the *cotangent plane* and denoted by $T_x^*\mathcal{M}$. To a vector field we can then associate a covector at each corresponding point, this defines a covector field, which is more classically called a *differential form* or a *1-form*. In the same way as the tangent bundle, one can define the *cotangent bundle* by

$$T^*\mathcal{M} = \{(\mathbf{x}, \mathbf{p}), \quad \mathbf{x} \in \mathcal{M}, \mathbf{p} \in T_x^*\mathcal{M}\}.$$

The natural basis of $T_x^*\mathcal{M}$ is the dual basis of $\boldsymbol{\partial}_1, \ldots, \boldsymbol{\partial}_n$ denoted by $\mathrm{d}q^1, \ldots, \mathrm{d}q^n$ or $\mathrm{d}^1, \ldots, \mathrm{d}^n$ and defined by $\mathrm{d}^i(\boldsymbol{\partial}_j) = \delta^i_j = 1$ if $i = j$ and 0 else. From this definition, as the $\boldsymbol{\partial}_j$ are the columns of $D\mathbf{F}$ and $(D\mathbf{F})^{-1}D\mathbf{F} = \mathbb{I}_n$, the d^i are the lines of $(D\mathbf{F})^{-1}$. So the components $\mathbf{p}^\top = (p_1, \ldots, p_n)$ of a covector p, which is a geometric object independent of the basis are defined by

$$\mathsf{p} = \sum_{i=1}^n p_i\,\mathrm{d}^i = \mathbf{p}^\top (D\mathbf{F})^{-1}.$$

The components of a covector define a line vector, which multiplies the inverse Jacobian matrix on the left. We therefore denote them using a transpose: $\mathbf{p}^\top$, and their components are denoted classically with a lower index. We can now express in local coordinates how a covector is applied to a vector:

$$\mathsf{p}(\mathsf{v}) = \sum_{i=1}^n p_i\,\mathrm{d}^i(\sum_{j=1}^n v^j \boldsymbol{\partial}_j) = \sum_{i=1}^n p_i v^i = \mathbf{p}^\top (D\mathbf{F})^{-1}(D\mathbf{F})\mathbf{v} = \mathbf{p}^\top \mathbf{v}.$$

We notice that the definition of the covectors is such that their application to a vector can be expressed directly using any local coordinate system without using the mapping $\mathbf{F}$. Indeed $\mathsf{p}(\mathsf{v}) = \mathbf{p}^\top \mathbf{v} = \mathbf{p}'^\top \mathbf{v}'$ independently of the local coordinate system. Because such sums involving a vector and a covector arrive frequently, we will use the Einstein sommation convention denoting by

$$p_i v^i = \sum_{i=1}^{n} p_i v^i.$$

A sum is implied where the same index appears as a lower index and an upper index in a product, hence implying a duality pairing between a covector and a vector.

p-forms. We have introduced 1-forms which associate a real number to any vector of the tangent plane $T_x\mathcal{M}$ at some given point. We shall use an index x to denote the dependence on a point on the manifold. We can now define the *tensor product* of two 1-forms as

$$\alpha_x \otimes \beta_x(\mathbf{v}, \mathbf{w}) = \alpha_x(\mathbf{v})\beta_x(\mathbf{w})$$

and the *exterior product* of two one forms as

$$\alpha_x \wedge \beta_x(\mathbf{v}, \mathbf{w}) = \alpha_x \otimes \beta_x(\mathbf{v}, \mathbf{w}) - \beta_x \otimes \alpha_x(\mathbf{v}, \mathbf{w}) = \alpha_x(\mathbf{v})\beta_x(\mathbf{w}) - \alpha_x(\mathbf{w})\beta_x(\mathbf{v}).$$

From this definition, it follows that the exterior product of two 1-forms is anti-symmetric

$$\alpha_x \wedge \beta_x(\mathbf{v}, \mathbf{w}) = -\beta_x \wedge \alpha_x(\mathbf{v}, \mathbf{w}).$$

The exterior product of two 1-forms should be a 2-form. So this leads to the definition of a 2-form as an antisymmetric element of $T_x^*\mathcal{M} \times T_x^*\mathcal{M}$ and more generally the p-forms are defined as follows:

Definition 5 *A p-form α^p is a p-linear form on $T_x\mathcal{M} \times \cdots \times T_x\mathcal{M}$ (p times), which is alternating, i.e.*

$$\begin{aligned} \alpha_x^p : \; & T_x\mathcal{M} \times \cdots \times T_x\mathcal{M} \to \mathbb{R} \\ & (\mathbf{v}_1, \ldots, \mathbf{v}_p) \mapsto \alpha_x^p(\mathbf{v}_1, \ldots, \mathbf{v}_p) \end{aligned}$$

such that

$$\alpha_x^p(\ldots, \mathbf{v}_i, \ldots, \mathbf{v}_j, \ldots) = -\alpha_x^p(\ldots, \mathbf{v}_j, \ldots, \mathbf{v}_i, \ldots).$$

When two vectors are exchanged, all the others staying the same, the sign of the p-form changes. This is a generalization of anti-symmetry to p variables. This implies in particular, that when a vector appears twice in the arguments the p-form vanishes. We denote a p-form by α^p with an upper right index when we wish to explicit the degree p of the form. When it is useful to recall its dependency on $\mathbf{x}$ we write α_x^p.

Due to the alternating property, in a manifold of dimension n all p-forms for $p > n$ vanish. This can be seen by decomposing all the vectors on a basis, which contains exactly n-elements.

We will denote the space of p-forms on the manifold $\mathcal{M}$ by $\Lambda^p(\mathcal{M})$ for $p = 0, \ldots, n$. The 0-forms are the scalar functions on $\mathcal{M}$. The n-forms in a manifold of dimension n are also well known from linear algebra. Indeed at a point x of the manifold a n-form is a n-covector defined as an alternating n-linear form, which is the definition of the determinant of n vectors.

Definition 6 *The* exterior *product of a k-form and a l-form is a mapping*

$$\wedge : \ \Lambda^k(\mathcal{M}) \times \Lambda^l(\mathcal{M}) \to \Lambda^{k+l}(\mathcal{M}) \qquad k + l \leq n$$

satisfying the following properties

$$\textit{antisymmetry:} \qquad \alpha^k \wedge \beta^l = (-1)^{kl} \beta^l \wedge \alpha^k \tag{2.1}$$

$$\textit{associativity:} \qquad (\alpha^k \wedge \beta^l) \wedge \gamma^m = \alpha^k \wedge (\beta^l \wedge \gamma^m) \tag{2.2}$$

$$\textit{distributivity:} \qquad (\alpha^k + \beta^k) \wedge \gamma^m = \alpha^k \wedge \gamma^m + \beta^k \wedge \gamma^m \tag{2.3}$$

where $\alpha^k \in \Lambda^k(\mathcal{M})$, $\beta^l \in \Lambda^l(\mathcal{M})$, $\gamma^m \in \Lambda^m(\mathcal{M})$.

Note that the wedge product with a 0-form is just a multiplication with a scalar function.

Given a basis $(\,\mathrm{d}q^1, \ldots, \mathrm{d}q^n)$ of the space of 1-forms $\Lambda^1(\mathcal{M})$, a basis of $\Lambda^p(\mathcal{M})$ is given by all possible combinations of p distinct $\mathrm{d}q^i$ by wedge product. The order is fixed arbitrarily, for example by imposing that the indices are in increasing order:

$$\alpha^p(\mathsf{x}) = \sum_{i_1 < \cdots < i_p} \alpha_{i_1, \ldots, i_p}(\mathsf{x})\, \mathrm{d}q^{i_1} \wedge \cdots \wedge \mathrm{d}q^{i_p}.$$

The dimension of $\Lambda^p(\mathcal{M})$ is thus $\binom{n}{p} = n!/(p!(n-p)!)$

Example: In $\mathbb{R}^3$, the zero forms are the scalar functions. The 1-forms, 2-forms, 3-forms can be expressed in the local basis by

$$\alpha^0(\mathsf{x}) = \alpha(\mathsf{x}), \tag{2.4}$$

$$\alpha^1(\mathsf{x}) = \alpha_1(\mathsf{x})\, \mathrm{d}q^1 + \alpha_2(\mathsf{x})\, \mathrm{d}q^2 + \alpha_3(\mathsf{x})\, \mathrm{d}q^3, \tag{2.5}$$

$$\alpha^2(\mathsf{x}) = \alpha_{23}(\mathsf{x})\, \mathrm{d}q^2 \wedge \mathrm{d}q^3 + \alpha_{31}(\mathsf{x})\, \mathrm{d}q^3 \wedge \mathrm{d}q^1 + \alpha_{12}(\mathsf{x})\, \mathrm{d}q^1 \wedge \mathrm{d}q^2, \tag{2.6}$$

$$\alpha^3(\mathsf{x}) = \alpha_{123}(\mathsf{x})\, \mathrm{d}q^1 \wedge \mathrm{d}q^2 \wedge \mathrm{d}q^3. \tag{2.7}$$

We note that the 1-forms and 2-forms have 3 scalar components and the 0-forms and 3-forms have 1 scalar component. In order to evaluate the p-form on vectors, one can use the formulas for determinants

$$\mathrm{d}q^i(\mathbf{v}) = v^i$$

$$\mathrm{d}q^i \wedge \mathrm{d}q^j(\mathbf{v},\mathbf{w}) = \begin{vmatrix} \mathrm{d}q^i(\mathbf{v}) & \mathrm{d}q^i(\mathbf{w}) \\ \mathrm{d}q^j(\mathbf{v}) & \mathrm{d}q^j(\mathbf{w}) \end{vmatrix} = v^i w^j - v^j w^i,$$

$$\mathrm{d}q^1 \wedge \mathrm{d}q^2 \wedge \mathrm{d}q^3(\mathbf{u},\mathbf{v},\mathbf{w}) = \begin{vmatrix} \mathrm{d}q^1(\mathbf{u}) & \mathrm{d}q^1(\mathbf{v}) & \mathrm{d}q^1(\mathbf{w}) \\ \mathrm{d}q^2(\mathbf{u}) & \mathrm{d}q^2(\mathbf{v}) & \mathrm{d}q^2(\mathbf{w}) \\ \mathrm{d}q^3(\mathbf{u}) & \mathrm{d}q^3(\mathbf{v}) & \mathrm{d}q^3(\mathbf{w}) \end{vmatrix} = \begin{vmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \end{vmatrix}.$$

Remark 9 *Following the intuitive definition of the Riemann integral, paving the space into small segments, we observe that 1-forms, which are applied to one single vector, will be naturally integrated along curves, 2-forms, which take 2 vectors, will be naturally integrated on surfaces and 3-forms taking 3 vectors will be naturally integrated on volumes. 0-forms for their part cannot be integrated, one only takes their point values.*

This idea comes back when discretising on a mesh. 0-form degrees of freedom are associated to points of the mesh (e.g. vertices), 1-forms to edges, 2-forms to faces and 3-forms to cells or volumes.

Riemanian metric The structures that we have build on our manifold up to now do not allow us to define length of vectors. For this we need a new structure call Riemannian metric defined as follows:

Definition 7 *A* Riemannian metric *g on a smooth manifold $\mathcal{M}$ is a scalar product (a bilinear symmetric positive definite form) on the tangent spaces $T_x\mathcal{M}$ that varies smoothly with $\mathbf{x}$, i.e.*

$$g_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$$

such that g_x is bilinear and

(i) $g_x(\mathbf{v},\mathbf{w}) = g_x(\mathbf{w},\mathbf{v})$, $\forall \mathbf{v},\mathbf{w} \in T_x\mathcal{M}$,

(ii) $g_x(\mathbf{v},\mathbf{v}) \geq 0$ $\forall \mathbf{v} \in T_x\mathcal{M}$,

(iii) $g_x(\mathbf{v},\mathbf{v}) = 0 \Leftrightarrow \mathbf{v} = 0$,

(iv) For any smooth vector fields $\mathbf{v}_x, \mathbf{w}_x$, the mapping $\mathbf{x} \mapsto g_x(\mathbf{v}_x, \mathbf{w}_x)$ is smooth.

In our applications, the tangent planes are $\mathbb{R}^n$, which is equipped with the euclidian scalar product $\cdot$. We can express it in the natural basis of $T_x\mathcal{M}$, $\boldsymbol{\partial}_1, \ldots, \boldsymbol{\partial}_n$:

$$g_x(\mathbf{v},\mathbf{w}) = \mathsf{v} \cdot \mathsf{w} = \sum_{i,j=1}^{n} v^i w^j \boldsymbol{\partial}_i \cdot \boldsymbol{\partial}_j.$$

Denoting by $g_{ij} = \boldsymbol{\partial}_i \cdot \boldsymbol{\partial}_j$, the scalar product is expressed in the tangent basis $\boldsymbol{\partial}_1, \ldots, \boldsymbol{\partial}_n$ by

$$g_x(\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{n} v^i w^j g_{ij} = v^i w^j g_{ij}$$

using the Einstein summation convention. Obviously this reduces to the classical formula $\sum (v^i)^2$ when the canonical basis of $\mathbb{R}^n$ is used (corresponding to an identity mapping) as $g_{ij} = \delta_{ij}$ in this case.

The metric enables us to associate a covector to a vector in a unique way. Indeed, given a vector $\mathbf{v} \in T_x\mathcal{M}$, $\mathbf{w} \mapsto g_x(\mathbf{v}, \mathbf{w})$ defines a linear form on $T_x\mathcal{M}$, i.e. an element $\mathsf{p} \in T_x^*\mathcal{M}$ such that $\mathbf{p}(\mathbf{w}) = g_x(\mathbf{v}, \mathbf{w})$. This can be written in local coordinates

$$\mathbf{p}^\top \mathbf{w} = \mathbf{v}^\top G \mathbf{w} \quad \text{or equivalently} \quad p_j w^j = v^i g_{ij} w^j$$

using Einstein's summation convention, that we will use systematically in the sequel, where G is the $n \times n$ matrix with entries g_{ij}. As this is verified for any $\mathbf{w} \in T_x\mathcal{M}$, it follows that $p_i = g_{ij} v^j$ ($\Leftrightarrow \mathbf{p}^\top = \mathbf{v}^\top G$). In particular in cartesian coordinates, where G is the identity matrix, a covector has the same components as the vector it is associated to. Note that due to the definition of g_{ij} with the tangent vectors, G can be expressed using the Jacobian of the mapping $G = (D\mathbf{F})^\top D\mathbf{F}$ and the inverse of G whose coefficients are denoted by g^{ij} verifies $G^{-1} = (D\mathbf{F})^{-1} D\mathbf{F}^{-\top}$, so that $g^{ij} = \mathbf{d}^i \cdot \mathbf{d}^j$.

Exterior derivative

Definition 8 *The* exterior derivative*, denoted by* d *is the unique operator from* $\Lambda^p(^{\mathcal{M}}) \to \Lambda^{p+1}(^{\mathcal{M}})$ *satisfying:*

(i) d *is additive:* $\mathrm{d}(\alpha + \beta) = \mathrm{d}\alpha + \mathrm{d}\beta$,

(ii) $\mathrm{d}\alpha^0$ *is the usual differential of functions,*

(iii) $\mathrm{d}(\alpha^p \wedge \beta^q) = \mathrm{d}\alpha^p \wedge \beta^q + (-1)^p \alpha^p \wedge \mathrm{d}\beta^q$*, (Leibnitz rule),*

(iv) $\mathrm{d}^2\alpha = \mathrm{d}(\mathrm{d}\alpha) = 0$.

See Frankel [11] p. 53 for a proof of the uniqueness.

Let us compute exterior derivatives in $\mathbb{R}^3$. For this we shall in particular need $\mathrm{d}\,\mathrm{d}q^i = 0$ and (iii) for a scalar function, which is a 0-form ($p = 0$):

$$\mathrm{d}(\alpha^0 \wedge \beta^q) = \mathrm{d}\alpha^0 \wedge \beta^q + \alpha^0\, \mathrm{d}\beta^q.$$

For a 0-form $\alpha_x^0 = \alpha(\mathbf{q})$ for some smooth scalar function α, we get

$$\mathrm{d}\alpha_x^0 = \frac{\partial \alpha}{\partial q^1}(\mathbf{q})\, \mathrm{d}q^1 + \frac{\partial \alpha}{\partial q^2}(\mathbf{q})\, \mathrm{d}q^2 + \frac{\partial \alpha}{\partial q^3}(\mathbf{q})\, \mathrm{d}q^3.$$

For a 1-form $\alpha_{\mathsf{x}}^1 = \alpha_1(\mathbf{q})\, \mathrm{d}q^1 + \alpha_2(\mathbf{q})\, \mathrm{d}q^2 + \alpha_3(\mathsf{x})\, \mathrm{d}q^3$, we have

$$\begin{aligned}
\mathrm{d}\alpha_{\mathsf{x}}^{(1)} &= \mathrm{d}\alpha_1(\mathbf{q}) \wedge \mathrm{d}q^1 + \mathrm{d}\alpha_2(\mathbf{q}) \wedge \mathrm{d}q^2 + \mathrm{d}\alpha_3(\mathsf{x}) \wedge \mathrm{d}q^3 \\
&= \sum_{i=1}^{3} \left(\frac{\partial \alpha_i}{\partial q^1}(\mathbf{q})\, \mathrm{d}q^1 + \frac{\partial \alpha_i}{\partial q^2}(\mathbf{q})\, \mathrm{d}q^2 + \frac{\partial \alpha_i}{\partial q^3}(\mathbf{q})\, \mathrm{d}q^3 \right) \wedge \mathrm{d}q^i \\
&= \left(\frac{\partial \alpha_2}{\partial q^1} - \frac{\partial \alpha_1}{\partial q^2} \right) \mathrm{d}q^1 \wedge \mathrm{d}q^2 + \left(\frac{\partial \alpha_1}{\partial q^3} - \frac{\partial \alpha_3}{\partial q^1} \right) \mathrm{d}q^3 \wedge \mathrm{d}q^1 \\
&\quad + \left(\frac{\partial \alpha_3}{\partial q^2} - \frac{\partial \alpha_2}{\partial q^3} \right) \mathrm{d}q^2 \wedge \mathrm{d}q^3
\end{aligned}$$

Note that the components of the exterior derivative of the 1-form are the components of the curl in cartesian coordinates.

For a 2-form $\alpha^{(2)}(\mathsf{x}) = \alpha_{23}(\mathbf{q})\,\mathrm{d}q^2 \wedge \mathrm{d}q^3 + \alpha_{31}(\mathbf{q})\,\mathrm{d}q^3 \wedge \mathrm{d}q^1 + \alpha_{12}(\mathbf{q})\,\mathrm{d}q^1 \wedge \mathrm{d}q^2$, we have

$$\begin{aligned}\mathrm{d}\alpha^{(2)}(\mathsf{x}) &= \mathrm{d}\alpha_{23} \wedge \mathrm{d}q^2 \wedge \mathrm{d}q^3 + \mathrm{d}\alpha_{31} \wedge \mathrm{d}q^3 \wedge \mathrm{d}q^1 + \mathrm{d}\alpha_{12} \wedge \mathrm{d}q^1 \wedge \mathrm{d}q^2 \\ &= \left(\frac{\partial \alpha_{23}}{\partial q^1} + \frac{\partial \alpha_{31}}{\partial q^2} + \frac{\partial \alpha_{12}}{\partial q^3}\right) \mathrm{d}q^1 \wedge \mathrm{d}q^2 \wedge \mathrm{d}q^3.\end{aligned}$$

Here we recognize the formula for the divergence in cartesian coordinates. And finally for a 3-form $\alpha^{(3)}(\mathsf{x}) = \alpha_{123}(\mathbf{q})\,\mathrm{d}q^1 \wedge \mathrm{d}q^2 \wedge \mathrm{d}q^3$, we get $\mathrm{d}\alpha^{(3)} = 0$.

Interior product

Definition 9 *The* interior product *of a p-form α^p and a vector $\mathbf{v}$ is the $(p-1)$-form denoted by $\mathrm{i}_{\mathbf{v}}\alpha^p$ such that*

$$\mathrm{i}_{\mathbf{v}}\alpha^p(\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}) = \alpha^p(\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{p-1}). \tag{2.8}$$

For a 0-form $\mathrm{i}_{\mathbf{v}}\alpha^0 = 0$.

Proposition 6 *The interior product $\mathrm{i}_{\mathbf{v}} : \Lambda^p(\mathcal{M}) \to \Lambda^{p-1}(\mathcal{M})$ is an* anti-derivation*, i.e. it verifies the Leibnitz rule*

$$\mathrm{i}_{\mathbf{v}}(\alpha^p \wedge \beta^q) = (\mathrm{i}_{\mathbf{v}}\alpha^p) \wedge \beta^q + (-1)^p \alpha^p \wedge \beta^q. \tag{2.9}$$

Note that this is the same property as property (*iii*) of the exterior derivative which is also an anti-derivation. The difference between an anti-derivation and a standard derivation (property of derivatives) is the $(-1)^p$ factor in front of the second term.

Orientation and twisted forms In an n-dimensional vector space two bases are related by a transformation matrix P such that, $\det P \neq 0$. From that one can define two classes of bases, those that have the same orientation are related by a transformation matrix with a positive sign and those that have a different orientation by a transformation matrix with a negative sign. The orientation of a vector space is chosen by arbitrarily choosing a basis, which defines the positive orientation. This is classically done in $\mathbb{R}^3$ with the right-hand rule.

Definition 10 *We say that a n-dimensional manifold $\mathbf{M}_n$ is orientable, if it can be covered with coordinate patches having all the same orientation.*

A classical example of non orientable manifold is the moebius band.

We shall denote by $o(\boldsymbol{\partial}_1, \ldots, \boldsymbol{\partial}_n)$ the orientation of a basis defined by a local coordinate system, $o(\boldsymbol{\partial}_1, \ldots, \boldsymbol{\partial}_n)$ can take the values $+1$ or -1.

The orientation is used to define *twisted forms* sometimes also called *pseudo-forms*, which are defined not to depend on the orientation of the local coordinate

system. To achieve this, the orientation is part of the definition of the twisted-form: In a positively oriented local coordinate system $\tilde{\alpha}^p = \alpha^p$ and in a negatively oriented local coordinate system $\tilde{\alpha}^p = -\alpha^p$.

An important twisted form is the *volume form*, which is a twisted n-form represented in a local coordinate system $(q^1, \ldots, q^n)$ on a riemannian manifold of dimension n by

$$\mathrm{vol}^n = o(\boldsymbol{\partial}_1, \ldots \boldsymbol{\partial}_n)\sqrt{g}\, \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n,$$

where $\sqrt{g} = \det(D\mathbf{F})$ is the Jacobian of the mapping defining the local coordinate system. Indeed as $G = (D\mathbf{F})^\top D\mathbf{F}$, $g := \det G = (\det(D\mathbf{F}))^2$.

Remark 10 *As the metric can be used to associate a 1-form to a vector, the volume form along with the interior product defines a (n-1)-form associated to a vector: Given a vector* $\mathbf{v}$ $\mathrm{i}_{\mathbf{v}}\mathrm{vol}^n$ *defines a (n-1)-form.*

In particular in a three dimensional manifold for a vector $\mathbf{u}$

$$\mathrm{i}_{\mathbf{u}}\mathrm{vol}^3(\mathbf{v}, \mathbf{w}) = \mathrm{vol}^3(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sqrt{g} \begin{vmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \end{vmatrix}$$

Hodge $\star$ operator and inner product of p-forms Given a metric g on an n-dimensional manifold $\mathcal{M}$ and a p-form

$$\alpha^p = \sum_{k_1 < \cdots < k_p} \alpha_{k_1 \ldots k_p}\, \mathrm{d}q^{k_1} \wedge \cdots \wedge \mathrm{d}q^{k_p}$$

we define the Hodge $\star$ of α^p at x and denote by $\star\alpha^p_x$ the pseudo $(n-p)$-form

$$\star\alpha^p_x = \tilde{\alpha}^{n-p}_x = \sum_{j_1 < \cdots < j_{n-p}} \alpha^*_{j_1 \ldots j_{n-p}}\, \mathrm{d}q^{j_1} \wedge \cdots \wedge \mathrm{d}q^{j_{n-p}},$$

with $\alpha^*_{j_1 \ldots j_{n-p}} = \sqrt{|g|}\alpha^{k_1 \ldots k_p}\epsilon_{k_1 \ldots k_p j_1 \ldots j_{n-p}}$ and where $\epsilon_{i_1 \ldots i_n}$ assumes each index appear only once and $\epsilon_{i_1 \ldots i_n} = 1$ for an even permutation of $(1, \ldots, n)$ and $\epsilon_{i_1 \ldots i_n} = -1$ for an odd permutation. Moreover $\alpha^{k_1 \ldots k_p} = \sum_{i_1, \ldots, i_p} g^{k_1 i_1} \ldots g^{k_p i_p} \alpha_{i_1 \ldots i_p}$. As an example, let us compute the Hodge $\star$ of general 1-forms and 2-forms in cartesian coordinates:

$$\star(\alpha_1\, \mathrm{d}q^1 + \alpha_2\, \mathrm{d}q^2 + \alpha_3\, \mathrm{d}q^3) = \alpha_1\, \mathrm{d}q^2 \wedge \mathrm{d}q^3 + \alpha_2\, \mathrm{d}q^3 \wedge \mathrm{d}q^1 + \alpha_3\, \mathrm{d}q^1 \wedge \mathrm{d}q^2.$$

An important example is the Hodge of the constant 0-form 1, which is the twisted volume form vol^n:

$$\star 1 = \sqrt{|g|}\epsilon_{12 \ldots n}\, \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n = \mathrm{vol}^n.$$

Note also that the Hodge $\star$ of a twisted p-form defines a (n-p)-form.

Let us also recall that applying twice the Hogde operator to a p-form in a manifold of dimension n yields

$$\star \star \alpha^p = (-1)^{p(n-p)}\alpha^p. \tag{2.10}$$

We observe that for n odd, in particular for $n = 3$, we always have $\star\star\,\alpha^p = \alpha^p$.

From this definition of the Hodge $\star$ it follows that for two p-forms α^p and β^p

$$\alpha^p \wedge \star\beta^p = \sum_{k_1<\cdots<k_p} \alpha_{k_1\ldots k_p}\beta^{k_1\ldots k_p}\mathrm{vol}^n = \langle\alpha^p, \beta^p\rangle\mathrm{vol}^n$$

where we define the scalar product of two p-forms at x on $T_x\mathcal{M}$ by

$$\langle\alpha^p_x, \beta^p_x\rangle = \sum_{k_1<\cdots<k_p} \alpha_{k_1\ldots k_p}\beta^{k_1\ldots k_p}.$$

Integrating over the manifold we can then define the following scalar product on $\Lambda^p(\mathcal{M})$

$$(\alpha^p, \beta^p) = \int \langle\alpha^p_x, \beta^p_x\rangle\mathrm{vol}^n = \int \alpha^p \wedge \star\beta^p. \tag{2.11}$$

Given this scalar product and its associated norm we define the Hilbert spaces

$$L^2\Lambda^p(\mathcal{M}) = \{\alpha^p \in \Lambda^p(\mathcal{M}) \mid (\alpha^p, \alpha^p) < +\infty\},$$

$$H\Lambda^p(\mathcal{M}) = \{\omega^p \in L^2\Lambda^p(\mathcal{M}),\, \mathrm{d}\omega^p \in L^2\Lambda^{p+1}(\Omega)\}.$$

These Hilbert spaces of differential forms form the so-called de Rham complex, which is at the heart of the FEEC theory

$$0 \to H\Lambda^0(\mathcal{M}) \overset{\mathrm{d}}{\to} H\Lambda^1(\mathcal{M}) \overset{\mathrm{d}}{\to} \cdots \overset{\mathrm{d}}{\to} H\Lambda^n(\mathcal{M}) \to 0. \tag{2.12}$$

In a three dimensional manifold the de Rham complex can also be express in the classical vector analysis language:

$$0 \to H^1(\Omega) \overset{\mathrm{grad}}{\to} H(\mathrm{curl}, \Omega) \overset{\mathrm{curl}}{\to} H(\mathrm{div}, \Omega) \overset{\mathrm{div}}{\to} L^2(\Omega) \to 0.$$

Pullback of differential forms Consider a smooth math $\boldsymbol{\varphi} : \mathcal{M} \to \mathcal{M}'$, where $\mathcal{M}$ and $\mathcal{M}'$ are two manifolds, not necessarily of the same dimension.

Through this map a vector in the tangent plane $\mathsf{v} \in T_x\mathcal{M}$ induces a vector $\boldsymbol{\varphi}_*\mathsf{v} = (D\boldsymbol{\varphi})\mathbf{v} \in T_{\boldsymbol{\varphi}(\mathsf{x})}\mathcal{M}'$ in the tangent plane at $\boldsymbol{\varphi}(\mathsf{x})$ of $\mathcal{M}'$ called the *push-forward* of v. From this one can for a p-form $\omega_{x'}$ at a point $\mathsf{x}' \in \mathcal{M}'$ a p-form at the pre-image of x' on $\mathcal{M}$ denote by $\boldsymbol{\varphi}^*\omega$ and called *pull-back* of the p-form ω. It is defined by

$$(\boldsymbol{\varphi}^*\omega)_x(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \omega_{\boldsymbol{\varphi}(\mathsf{x})}((D\boldsymbol{\varphi})_x\mathbf{v}_1, \ldots, (D\boldsymbol{\varphi})_x\mathbf{v}_p). \tag{2.13}$$

Very important and convenient properties of the pullback are that they respect exterior products and differentiation

$$\boldsymbol{\varphi}^*(\alpha \wedge \beta) = (\boldsymbol{\varphi}^*\alpha) \wedge (\boldsymbol{\varphi}^*\beta), \qquad \mathrm{d}(\boldsymbol{\varphi}^*\alpha) = \boldsymbol{\varphi}^*(\,\mathrm{d}\alpha). \tag{2.14}$$

An important special case is when $\mathcal{M}'$ is a submanifold of $\mathcal{M}$ and $\boldsymbol{\varphi}$ the inclusion, then the pullback is the trace map. With this definition we ca write the Stokes theorem

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \mathrm{tr}\,\omega.$$

Green formula and codifferential operator Using the Leibnitz rule, (iii) of definition 8, and the Stokes theorem, we can derive a Green type integration by parts formula for differential forms

$$\int_{\mathcal{M}} \mathrm{d}\alpha^{p-1} \wedge \beta^{n-p} = (-1)^p \int_{\mathcal{M}} \alpha^{p-1} \wedge \mathrm{d}\beta^{n-p} + \int_{\partial\mathcal{M}} \mathrm{tr}\, \alpha^{p-1} \wedge \mathrm{tr}\, \beta^{n-p},$$
$$\forall \alpha^{p-1} \in \Lambda^{p-1}(\mathcal{M}),\ \beta^{n-p} \in \Lambda^{n-p}(\mathcal{M}). \quad (2.15)$$

Let us now introduce the coderivative operator $\mathrm{d}^* : \Lambda^p(\mathcal{M}) \to \Lambda^{p-1}(\mathcal{M})$ defined by

$$\star \mathrm{d}^* \alpha^p = (-1)^p \, \mathrm{d} \star \alpha^p. \quad (2.16)$$

We can then express the Green formula (2.15) using the scalar product on differential forms (2.11). Indeed setting $\beta^{n-p} = \star\gamma^p$, (2.15) becomes

$$\int_{\mathcal{M}} \mathrm{d}\alpha^{p-1} \wedge \star\gamma^p = (-1)^p \int_{\mathcal{M}} \alpha^{p-1} \wedge \mathrm{d}\star\gamma^p + \int_{\partial\mathcal{M}} \mathrm{tr}\, \alpha^{p-1} \wedge \mathrm{tr} \star\gamma^p,$$

and so using the definition of the codifferential and of the scalar product (2.11) this can be written equivalently

$$(\mathrm{d}\alpha^{p-1}, \gamma^p) = (\alpha^{p-1}, \mathrm{d}^*\gamma^p) + \int_{\partial\mathcal{M}} \mathrm{tr}\, \alpha^{p-1} \wedge \mathrm{tr} \star\gamma^p. \quad (2.17)$$

Hence, the codifferential is the formal adjoint of the exterior derivative. It is the adjoint if the boundary terms vanish. This explains the notation.

2.2 Maxwell's equations with differential forms

There is a natural interpretation of physics objects with differential forms. In many cases it is more natural than with vectors. For example a force is naturally defined through the work it performs along a curve, which leads to its being a 1-form. The magnetic intensity is measured as a flux through a loop and thus is naturally a 2-form.

In classical vector analysis, Maxwell's equations read

$$-\frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{H} = \mathbf{J}, \quad (2.18)$$
$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \quad (2.19)$$
$$\nabla \cdot \mathbf{D} = \rho, \quad (2.20)$$
$$\nabla \cdot \mathbf{B} = 0. \quad (2.21)$$

These relations are supplemented by the material constitutive relations $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, where ε is the permittivity tensor (in vacuum it is the constant ε_0), and μ is the permeability tensor of the material (in vacuum it is the constant μ_0). $\mathbf{H}$ and $\mathbf{E}$ are respectively the magnetic field and the electric field, whereas $\mathbf{B}$ is the called

magnetic flux intensity or magnetic induction and $\mathbf{D}$ is the electric displacement field.

Given that the curl corresponds to the exterior derivative applied to a 1-form and the divergence to the exterior derivative applied to a 2-form, we find a natural way of writing Maxwell's equations in terms of differential forms

$$-\frac{\partial \mathbf{d}^2}{\partial t} + \mathrm{d}\mathbf{h}^1 = \mathbf{j}^2 \tag{2.22}$$

$$\frac{\partial \mathbf{b}^2}{\partial t} + \mathrm{d}\mathbf{e}^1 = 0, \tag{2.23}$$

$$\mathrm{d}\mathbf{d}^2 = \rho^3, \tag{2.24}$$

$$\mathrm{d}\mathbf{b}^2 = 0. \tag{2.25}$$

The constitutive relations associate 2-forms to 1-forms and thus need to be described by a Hodge $\star$ operator in the differential forms description:

$$\mathbf{d}^2 = \star\mathbf{e}^1, \qquad \mathbf{h}^1 = \star\mathbf{b}^2. \tag{2.26}$$

As the Hodge operator associates twisted forms to straight forms and conversely, it follows from this that either the forms in Ampere's equations (2.22) are twisted forms and the forms in Faraday's equation (2.23) are straight forms other the other way. Mathematically both options are possible, but physically the sources ρ^3 and $\mathbf{j}^2$ are charge and current densities which should be twisted see e.g. [6, 10]. Therefore, we will consider that $\mathbf{h}^1$, $\mathbf{d}^2$, $\mathbf{j}^2$ and ρ^3 are twisted forms and $\mathbf{e}^1$ and $\mathbf{b}^2$ are straight forms. The electrostatic scalar potential ϕ and the vector potential $\mathbf{A}$ also play an important role in electromagnetism, they can be describe respectively by the 0-form ϕ^0 and the 1-form $\mathbf{a}^1$ and are then related to the electric field and magnetic induction by

$$\mathbf{e}^1 = -\mathrm{d}\phi^0 - \frac{\partial \mathbf{a}^1}{\partial t}, \qquad \mathbf{b}^2 = \mathrm{d}\mathbf{a}^1. \tag{2.27}$$

Note that taking the exterior derivative of the first equation yields Faraday's law (2.23):

$$\mathrm{d}\mathbf{e}^1 = -\mathrm{d}\mathrm{d}\phi^0 - \frac{\partial\, \mathrm{d}\mathbf{a}^1}{\partial t} = -\frac{\partial \mathbf{b}^2}{\partial t},$$

and taking the exterior derivative of the first equation yields (2.25)

$$\mathrm{d}\mathbf{b}^2 = \mathrm{d}\mathrm{d}\mathbf{a}^1 = 0.$$

Another important physical object related to Maxwell's equations is the Lorentz force involving $\mathbf{E} + \mathbf{v} \times \mathbf{B}$. The velocity vector being a *vector*, this quantity is naturally expressed in the language of differential forms as $\mathbf{e}^1 + \mathrm{i}_{\mathbf{v}}\mathbf{b}^2$.

Remark 11 *It is important to note here that the exterior derivative and the time derivative here are independent of the metric and thus Maxwell's equations* (2.22)-(2.25) *as well as the equation defining the potentials* (2.27) *have the same form in any coordinate system. The metric only appears in the Hodge operator* (2.26)*, which relates the 1-form and 2-form expressions of the electric and magnetic fields.*

A geometric discretisation should aim at preserving these properties. There are at least two classical strategies to discretise the geometric form of Maxwell's equations. The first one is based on Finite Difference in the framework of the so-called Discrete Exterior Calculus (DEC) [8]. In this framework dual grids are constructed, the straight forms being discretised on the primal grid and the twisted forms on the dual grid. The discrete Hodge operators defines a projection from one grid to the other. The second one is called Finite Element Exterior Calculus (FEEC) [4, 1]. It is constructed on compatible Finite Elements spaces for each degree of forms and a unique grid. The straight forms are approximated in a strong sense, yielding relations between the coefficients in the basis expressions of the discrete differential forms. The twisted forms are approximated in the same Finite Element spaces but in weak form. This can be naturally expressed using the codiferential. Indeed using (2.26), Ampere's equations can also be written

$$-\frac{\partial \star \mathbf{e}^1}{\partial t} + \mathrm{d}\star\mathbf{b}^2 = \mathbf{j}^2$$

and using that $\star\star = \mathrm{Id}$ in three dimensions

$$-\frac{\partial \mathbf{e}^1}{\partial t} + \mathrm{d}^*\mathbf{b}^2 = \star\mathbf{j}^2.$$

In the same way we can express (2.24) using $\mathbf{e}^1$: $\mathrm{d}\star\mathbf{e}^1 = \rho^3$ and using the expression of the codifferential for a 1-form

$$-\mathrm{d}^*\mathbf{e}^1 = \star\rho^3.$$

The expression of Maxwell's equations with differential forms suitable for Finite Element discretisation thus reads

$$-\frac{\partial \mathbf{e}^1}{\partial t} + \mathrm{d}^*\mathbf{b}^2 = \star\mathbf{j}^2 \tag{2.28}$$

$$\frac{\partial \mathbf{b}^2}{\partial t} + \mathrm{d}\mathbf{e}^1 = 0, \tag{2.29}$$

$$-\mathrm{d}^*\mathbf{e}^1 = \star\rho^3, \tag{2.30}$$

$$\mathrm{d}\mathbf{b}^2 = 0. \tag{2.31}$$

The two equations involving the codifferential can be expressed in weak form to replace the codifferential by the exterior derivative. Indeed taking the scalar product of (2.28) with a test 1-form $\mathbf{f}^1$ and using that d^* is the adjoint of d provided the boundary terms vanish (else they need to be added as required by (2.17))

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{e}^1, \mathbf{f}^1) + (\mathbf{b}^2, \mathrm{d}\mathbf{f}^1) = (\star\mathbf{j}^2, \mathbf{f}^1) \quad \forall \mathbf{f}^1 \in H\Lambda^1(\mathcal{M}). \tag{2.32}$$

We recognize here the weak form of Ampère's equation and in the same way taking the inner product of (2.30) with a test 0-form yields the weak form of Gauss's law:

$$-(\mathbf{e}^1, \mathrm{d}\psi^0) = (\star\rho^3, \psi^0) \quad \forall \psi^0 \in H\Lambda^0(\mathcal{M}). \tag{2.33}$$

2.3 Discrete differential forms based on B-splines

Some of the standard Finite Element spaces seen in the first part of the lecture can be put to together to define a complex of discrete differential forms. This is an essential piece needed in the FEEC theory to ensure stability and convergence of the Finite Element approximation. These complexes are described in [4, 1]. A classical complex based on polynomials of degree k for a mesh of tetrahedra is

$$0 \to \mathbb{P}_k(\Omega) \overset{\text{grad}}{\to} N_k(\Omega) \overset{\text{curl}}{\to} RT_k(\Omega) \overset{\text{div}}{\to} DG_{k-1}(\Omega) \to 0.$$

Let us now construct a different kind of Finite Element discretization using basis functions for the different spaces based on B-Splines.

In order to define a family of n B-splines of degree k, we need $(x_i)_{0\leqslant i\leqslant n+k}$ a non-decreasing sequence of points on the real line called *knots* in the spline terminology. There can be several knots at the same position. In the case when there are m knots at the same point, we say that the knot has multiplicity m.

Definition 11 (B-Spline) *Let $(x_i)_{0\leqslant i\leqslant n+k}$ be a non-decreasing sequence of knots. Then the j-th B-Spline ($0 \leq j \leq n-1$) denoted by N_j^k of degree k is defined by the recurrence relation:*

$$N_j^k(x) = w_j^k(x) N_j^{k-1}(x) + (1 - w_{j+1}^k(x)) N_{j+1}^{k-1}(x)$$

where,

$$w_j^k(x) = \frac{x - x_j}{x_{j+k} - x_j}, \qquad N_j^0(x) = \chi_{[x_j, x_{j+1}[}(x).$$

We note some important properties of a B-splines basis:

- B-splines are piecewise polynomial of degree k,
- B-splines are non negative,
- Compact support; the support of N_j^k is contained in $[t_j, .., t_{j+k+1}]$,
- Partition of unity: $\sum_{i=0}^{n-1} N_i^k(x) = 1, \forall x \in \mathbb{R}$,
- Local linear independence,
- If a knot x_i has a multiplicity m then the B-spline is $\mathcal{C}^{(k-m)}$ at x_i.

A key point for constructing a complex of Finite Element spaces for p-forms comes from the recursion formula for the derivatives:

$$N_i^{k\prime}(x) = k \left(\frac{N_i^{k-1}(x)}{x_{i+k} - x_i} - \frac{N_{i+1}^{k-1}(x)}{x_{i+k+1} - x_{i+1}} \right). \tag{2.34}$$

It will be convenient to introduce the notation $D_i^k(x) = k \frac{N_i^{k-1}(x)}{x_{i+k} - x_i}$. Then the recursion formula for derivatives simply becomes

$$N_i^{k\prime}(x) = D_i^k(x) - D_{i+1}^k(x). \tag{2.35}$$

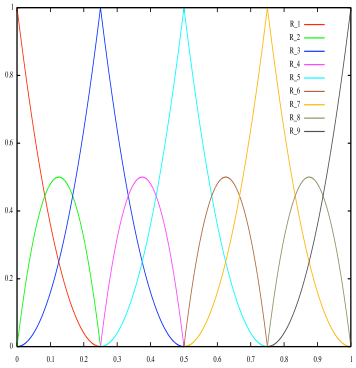

Figure 2.3: All B-splines functions associated to a knot sequence defined by $n = 9$, $k = 2$, $T = \{000, \frac{1}{4}\frac{1}{4}, \frac{1}{2}\frac{1}{2}, \frac{3}{4}\frac{3}{4}, 111\}$

Remark 12 *In the case where all knots, except the boundary knots are of multiplicity 1, the set $(N_i^k)_{0\leq i\leq n-1}$ of B-splines of degree k forms a basis of the spline space ($\dim \mathcal{S}^k = n$) defined by*

$$\mathcal{S}^k = \{v \in C^{k-1}([x_0, x_n]) \; | v_{|[x_i,x_{i+1}]} \in \mathbb{P}_k([x_i, x_{i+1}]).$$

The boundary knots are chosen to have multiplicity $k+1$ so that the spline becomes interpolatory on the boundary in order to simplify the application of Dirichlet boundary conditions. This setting is called open boundary conditions for the spline.

Then due to the definitions if follows immediately that $(D_i^k)_{1\leq i\leq n-1}$ is a basis of $\mathcal{S}^{k-1}$. Note that if the first knot has multiplicity $k+1$, D_0^k will have a support restricted to one point and be identically 0.

Remark 13 *Splines can be easily defined in the case of periodic boundary conditions by taking a periodic knot sequence. The dimension of the spline space is then the number of distinct knots.*

Let us construct spaces of differential forms whose coefficients are splines. In order to obtain a complex, we shall start with the space of 0-forms and then construct the space of p-forms by applying the exterior derivative to the $(p-1)$-forms. This property is needed to form a complex. Let us start with a finite dimensional space of 0-forms denoted by V_0.

We define the 3D point $\mathbf{x} = (x_1, x_2, x_3)$ and the multi-index $\mathbf{i} = (i_1, i_2, i_3)$. The the basis functions for 0-forms will be

$$\varphi_{\mathbf{i}}^0(\mathbf{x}) = N_{i_1}^k(x_1) N_{i_2}^k(x_2) N_{i_3}^k(x_3), \quad 1 \leq i_1 \leq n_1, 1 \leq i_2 \leq n_2, 1 \leq i_3 \leq n_3. \quad (2.36)$$

And our finite dimensional space of 0-forms base on B-splines of degree k in each direction is then $V_0 = \text{Span}\{\varphi_{\mathbf{i}}^0\}_{\mathbf{i}}$. And any discrete 0-form can be written as

$$\psi_h^0(\mathbf{x}) = \sum_{\mathbf{i}} c_{\mathbf{i}}^0 \varphi_{\mathbf{i}}^0(\mathbf{x}).$$

Now taking the exterior derivative of ϕ_h^0 we find

$$\begin{aligned}
\mathrm{d}\psi_h^0(\mathbf{x}) &= \sum_{\mathbf{i}} c_{\mathbf{i}}^0 \,\mathrm{d}\varphi_{\mathbf{i}}^0(\mathbf{x}) = \sum_{\mathbf{i}} c_{\mathbf{i}}^0((N_{i_1}^k)'(x_1)N_{i_2}^k(x_2)N_{i_3}^k(x_3)\,\mathrm{d}x^1 \\
&\quad + N_{i_1}(x_1)(N_{i_2}^k)'(x_2)N_{i_3}^k(x_3)\,\mathrm{d}x^2 + N_{i_1}^k(x_1)N_{i_2}^k(x_2)(N_{i_3}^k)'(x_3)\,\mathrm{d}x^3), \\
&= \sum_{\mathbf{i}} c_{\mathbf{i}}^0((D_{i_1}^k - D_{i_1+1}^k)(x_1)N_{i_2}^k(x_2)N_{i_3}^k(x_3)\,\mathrm{d}x^1 \\
&\quad + N_{i_1}(x_1)(D_{i_2}^k - D_{i_2+1}^k)(x_2)N_{i_3}^k(x_3)\,\mathrm{d}x^2 + N_{i_1}^k(x_1)N_{i_2}^k(x_2)(D_{i_3}^k - D_{i_3+1}^k)(x_3)\,\mathrm{d}x^3), \\
&= \sum_{\mathbf{i}} (c_{i_1,i_2,i_3}^0 - c_{i_1-1,i_2,i_3}^0)D_{i_1}^k(x^1)N_{i_2}^k(x_2)N_{i_3}^k(x_3)\,\mathrm{d}x^1 \\
&\quad + (c_{i_1,i_2,i_3}^0 - c_{i_1,i_2-1,i_3}^0)N_{i_1}^k(x^1)D_{i_2}^k(x_2)N_{i_3}^k(x_3)\,\mathrm{d}x^2 \\
&\quad + (c_{i_1,i_2,i_3}^0 - c_{i_1,i_2,i_3-1}^0)N_{i_1}^k(x^1)N_{i_2}^k(x_2)D_{i_3}^k(x_3)\,\mathrm{d}x^3,
\end{aligned}$$

where we used expression (2.35) for the derivative of a B-spline.

In order to form a complex the finite dimensional space of 1-forms V_1 should contain all exterior derivatives of 0-forms. This will be the case if we choose as basis functions for the 1-forms

$$\begin{aligned}
\boldsymbol{\varphi}_{\mathbf{i},1}^1(\mathbf{x}) &= D_{i_1}^k(x^1)N_{i_2}^k(x_2)N_{i_3}^k(x_3)\,\mathrm{d}x^1 = \varphi_{\mathbf{i},1}^1(\mathbf{x})\,\mathrm{d}x^1, \\
\boldsymbol{\varphi}_{\mathbf{i},2}^1(\mathbf{x}) &= N_{i_1}^k(x^1)D_{i_2}^k(x_2)N_{i_3}^k(x_3)\,\mathrm{d}x^2 = \varphi_{\mathbf{i},2}^1(\mathbf{x})\,\mathrm{d}x^2, \\
\boldsymbol{\varphi}_{\mathbf{i},3}^1(\mathbf{x}) &= N_{i_1}^k(x^1)N_{i_2}^k(x_2)D_{i_3}^k(x_3)\,\mathrm{d}x^3 = \varphi_{\mathbf{i},3}^1(\mathbf{x})\,\mathrm{d}x^3.
\end{aligned}$$

A general finite dimensional 1-form is then represented by

$$\mathbf{f}^1(\mathbf{x}) = \sum_{\mathbf{i}=1}^{\mathbf{n}} \sum_{a=1}^{3} c_{\mathbf{i},a}^1 \boldsymbol{\varphi}_{\mathbf{i},a}^1(\mathbf{x}) = \sum_{\mathbf{i}=1}^{\mathbf{n}} \sum_{a=1}^{3} c_{\mathbf{i},a}^1 \varphi_{\mathbf{i},a}^1(\mathbf{x})\,\mathrm{d}x^a.$$

In particular, if $\mathbf{f}^1 = \mathrm{d}\phi^0$, there coefficients in their respective bases are related by

$$c_{\mathbf{i},1}^1 = c_{i_1,i_2,i_3}^0 - c_{i_1-1,i_2,i_3}^0, \quad c_{\mathbf{i},2}^1 = c_{i_1,i_2,i_3}^0 - c_{i_1,i_2-1,i_3}^0, \quad c_{\mathbf{i},3}^1 = c_{i_1,i_2,i_3}^0 - c_{i_1,i_2,i_3-1}^0,$$

independently of the metric. It depends only on the neighboring splines.

In the same way, to construct a basis for V_2 we take the exterior derivative of an element of V_1. This yields

$$\begin{aligned}
\mathrm{d}\mathbf{f}^1(\mathbf{x}) &= \sum_{\mathbf{i}=1}^{\mathbf{n}} \sum_{a=1}^{3} c_{\mathbf{i},a}^1 \,\mathrm{d}\boldsymbol{\varphi}_{\mathbf{i},a}^1(\mathbf{x}) && (2.37) \\
&= \sum_{\mathbf{i}=1}^{\mathbf{n}} \big((c_{i_1,i_2,i_3,3}^1 - c_{i_1,i_2-1,i_3,3}^1) - (c_{i_1,i_2,i_3,2}^1 - c_{i_1,i_2,i_3-1,2}^1)\big) \\
&\qquad N_{i_1}^k(x^1)D_{i_2}^k(x_2)D_{i_3}^k(x_3)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 && (2.38) \\
&\quad + \big((c_{i_1,i_2,i_3,1}^1 - c_{i_1,i_2,i_3-1,1}^1) - (c_{i_1,i_2,i_3,3}^1 - c_{i_1-1,i_2,i_3,3}^1)\big) \\
&\qquad D_{i_1}^k(x^1)N_{i_2}^k(x_2)D_{i_3}^k(x_3)\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 && (2.39) \\
&\quad + \big((c_{i_1,i_2,i_3,2}^1 - c_{i_1-1,i_2,i_3,2}^1) - (c_{i_1,i_2,i_3,1}^1 - c_{i_1,i_2-1,i_3,1}^1)\big) \\
&\qquad D_{i_1}^k(x^1)D_{i_2}^k(x_2)N_{i_3}^k(x_3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 && (2.40)
\end{aligned}$$

In order to form a complex the space of finite dimensional 2-forms should contain all exterior derivatives of 1-forms. This will be the case if we choose as basis functions for the 2-forms

$$\begin{aligned}
\boldsymbol{\varphi}^2_{\mathbf{i},1}(\mathbf{x}) &= N^k_{i_1}(x^1)D^k_{i_2}(x_2)D^k_{i_3}(x_3)\,\mathrm{d}x^2\wedge\mathrm{d}x^3 = \varphi^2_{\mathbf{i},1}(\mathbf{x})\,\mathrm{d}x^2\wedge\mathrm{d}x^3,\\
\boldsymbol{\varphi}^2_{\mathbf{i},2}(\mathbf{x}) &= D^k_{i_1}(x^1)N^k_{i_2}(x_2)D^k_{i_3}(x_3)\,\mathrm{d}x^3\wedge\mathrm{d}x^1 = \varphi^2_{\mathbf{i},2}(\mathbf{x})\,\mathrm{d}x^3\wedge\mathrm{d}x^1,\\
\boldsymbol{\varphi}^1_{\mathbf{i},3}(\mathbf{x}) &= D^k_{i_1}(x^1)D^k_{i_2}(x_2)N^k_{i_3}(x_3)\,\mathrm{d}x^1\wedge\mathrm{d}x^2 = \varphi^2_{\mathbf{i},3}(\mathbf{x})\,\mathrm{d}x^1\wedge\mathrm{d}x^2.
\end{aligned}$$

A general discrete 2-form is then represented by

$$\mathbf{b}^2(\mathbf{x}) = \sum_{\mathbf{i}=1}^{\mathbf{n}}\sum_{a=1}^{3} c^2_{\mathbf{i},a}\boldsymbol{\varphi}^2_{\mathbf{i},a}(\mathbf{x}).$$

In particular, if $\mathbf{b}^2 = \mathrm{d}\mathbf{f}^1$, their coefficients in their respective bases are related by

$$\begin{aligned}
c^2_{\mathbf{i},1} &= (c^1_{i_1,i_2,i_3,3} - c^1_{i_1,i_2-1,i_3,3}) - (c^1_{i_1,i_2,i_3,2} - c^1_{i_1,i_2,i_3-1,2}),\\
c^2_{\mathbf{i},2} &= (c^1_{i_1,i_2,i_3,1} - c^1_{i_1,i_2,i_3-1,1}) - (c^1_{i_1,i_2,i_3,3} - c^1_{i_1-1,i_2,i_3,3}),\\
c^2_{\mathbf{i},3} &= (c^1_{i_1,i_2,i_3,2} - c^1_{i_1-1,i_2,i_3,2}) - (c^1_{i_1,i_2,i_3,1} - c^1_{i_1,i_2-1,i_3,1}),
\end{aligned}$$

independently of the metric. It depends only on the neighboring splines.

We finally come to the last space V_3 of discrete 3-forms. These are constructed such that the exterior derivative of an element of V_2 is in V_3. Taking the exterior derivative of a generic element $\mathbf{b}^2 \in V_2$ yields

$$\begin{aligned}
\mathbf{b}^2(\mathbf{x}) &= \sum_{\mathbf{i}=1}^{\mathbf{n}}\sum_{a=1}^{3} c^2_{\mathbf{i},a}\,\mathrm{d}\boldsymbol{\varphi}^2_{\mathbf{i},a}(\mathbf{x})\\
&= \big((c^2_{i_1,i_2,i_3,1} - c^2_{i_1-1,i_2,i_3,1}) + (c^2_{i_1,i_2,i_3,2} - c^2_{i_1,i_2-1,i_3,2}) + (c^2_{i_1,i_2,i_3,3} - c^2_{i_1,i_2,i_3-1,3})\big)\\
&\qquad D^k_{i_1}(x^1)D^k_{i_2}(x_2)D^k_{i_3}(x_3)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3.
\end{aligned}$$

So the basis functions for the three forms are of the form

$$\varphi^3_{\mathbf{i}}(\mathbf{x}) = D^k_{i_1}(x_1)D^k_{i_2}(x_2)D^k_{i_3}(x_3)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3,$$

and a general 3-form writes

$$\rho^3(\mathbf{x}) = \sum_{\mathbf{i}=1}^{\mathbf{n}} c^3_{\mathbf{i}}\varphi^3_{\mathbf{i}}(\mathbf{x}). \tag{2.41}$$

And the coefficients in V_3 of the exterior derivative of an element of V_2 are given by

$$c^3_{\mathbf{i}} = (c^2_{i_1,i_2,i_3,1} - c^2_{i_1-1,i_2,i_3,1}) + (c^2_{i_1,i_2,i_3,2} - c^2_{i_1,i_2-1,i_3,2}) + (c^2_{i_1,i_2,i_3,3} - c^2_{i_1,i_2,i_3-1,3}).$$

Here again this is independent of the metric an only involves neighboring spline coefficients.

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