B-Splines and IsoGeometric Analysis

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1 Introduction

2 Cardinal B-Splines

Cardinal B-Splines play an important role in the approximation theory (multi-resolution approximation, ...). In the sequel, we shall give a definition of the Cardinal B-Spline using the convolution operator. Then, we will present some of the most important properties, at least needed when using uniform B-Splines in a Finite Elements method.

Definition 2.1. *A cardinal B-spline of zero degree, denoted by ϕ_0, is the characteristic function over the interval* $[0, 1)$*, i.e.,*

$$\phi_0(t) := \begin{cases} 1, & t \in [0, 1) \\ 0, & \textit{otherwise} \end{cases} \tag{1}$$

A cardinal B-Spline of degree p, $p \in \mathbb{N}$, denoted by ϕ_p, is defined by convolution as

$$\phi_p(t) = \Big(\phi_{p-1} * \phi_0\Big)(t) = \int_{\mathbb{R}} \phi_{p-1}(t - s)\phi_0(s)\, ds \tag{2}$$

Example 1: When $p = 1$, it is easy to show that

$$\phi_1(t) := \begin{cases} t, & t \in [0, 1) \\ 2 - t, & t \in [1, 2) \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

When $p = 2$, it is easy to show that

$$\phi_2(t) := \begin{cases} \frac{1}{2}x^2, & t \in [0, 1) \\ \frac{1}{2} + (x - 1) - (x - 1)^2, & t \in [1, 2) \\ \frac{1}{2} - (x - 2) + \frac{1}{2}(x - 2)^2, & t \in [2, 3) \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

In figure (Fig. 1), we plot the Cardinal B-Splines of degrees $1, 2, 3$ and 4.

Remark 2.1. *The colored area under the graph of ϕ_2 represents the average $\int_{x-1}^{x} \phi_2(t)dt$ which is the value $\phi_3(x)$.*

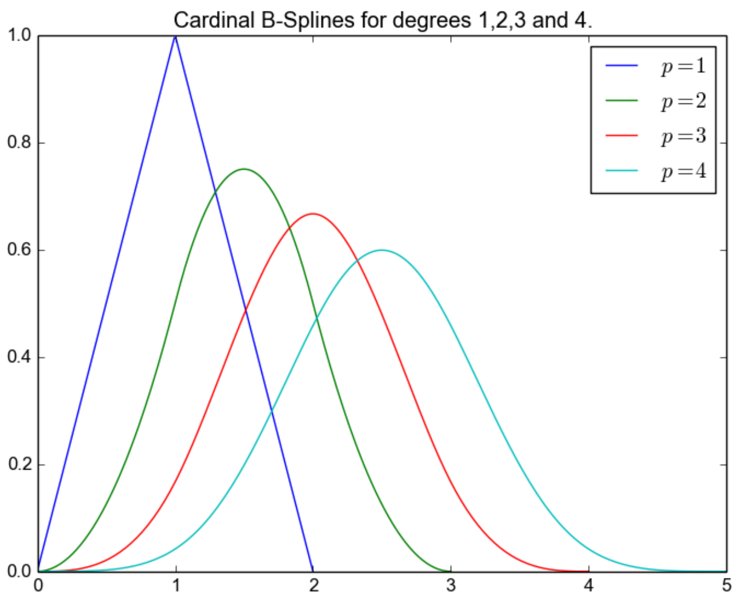

Figure 1: Cardinal B-Splines of degrees $1, 2, 3$ and 4

2.1 Cardinal B-Splines properties

Let ϕ_p be a cardinal B-Spline of degree p, $p \in \mathbb{N}$. The following properties can be proved by induction on the B-Spline degree p.

Theorem 2.2 (Minimal support)**.** *the support of ϕ_p is* $[0, p+1]$

Theorem 2.3 (Positivity)**.** $\phi_p(s) \geq 0, \forall s \in [0, p+1]$

Theorem 2.4. $\phi_p \in C^{p-1}$

Theorem 2.5. ϕ_p *is a piecewise-polynomial of degree p at each interval* $[i, i+1], \forall i \in \{0, 1, \ldots, p\}$

The sequence $\{0, 1, 2, \ldots, p\}$ is known as the *breaks* of the cardinal B-Spline of degree p.

Theorem 2.6. $\forall t \in [0, p+1]$ *and* $p \geq 1$*, we have*

$$\dot{\phi}_p(t) = \phi_{p-1}(t) - \phi_{p-1}(t-1) \tag{5}$$

Theorem 2.7 (Symmetry)**.** ϕ_p *is symmetric on the interval* $[0, p+1]$*, i.e.*

$$\phi_p(t) = \phi_p(p+1-t), \quad \forall t \in [0, p+1] \tag{6}$$

The following theorem was proved in 1972 by both Cox and Deboor separatly.

Theorem 2.8 (Cox-Deboor)**.** $\forall t \in [0, p+1]$ *and* $p \geq 1$*, we have*

$$\phi_p(t) = \frac{t}{p}\phi_{p-1}(t) + \frac{p+1-t}{p}\phi_{p-1}(t-1) \tag{7}$$

Proof: Let $\phi_i^p(t) := \phi_p(t-i), \forall t \in [0, p+1]$. We will proof the result by induction. Since both sides vanish at $t = 0$, we will use the equivalence to the formula for the derivative 2.6.

$$\phi_0^{p-1} - \phi_1^{p-1} = \frac{1}{p}\left(\phi_0^{p-1} - \phi_1^{p-1}\right) + \left[\frac{t}{p}\left(\phi_0^{p-2} - \phi_1^{p-2}\right) + \frac{p+1-t}{p}\left(\phi_1^{p-2} - \phi_2^{p-2}\right)\right] \tag{8}$$

The last term of the previous relation, can be written as

$$\frac{p-1}{p}\left[\left(\frac{t}{p-1}\phi_0^{p-2} + \frac{p-t}{p-1}\phi_1^{p-2}\right) - \left(\frac{t-1}{p-1}\phi_1^{p-2} + \frac{p-(t-1)}{p-1}\phi_2^{p-2}\right)\right] \tag{9}$$

Now, if we assume that the recursion is valid up to $p-1$, then the last terms is equal to

$$\frac{p-1}{p}\left(\phi_0^{p-1} - \phi_1^{p-1}\right) \tag{10}$$

□

For any Cardinal B-Spline of degree p, we denote by $\alpha_i^p = \left(\alpha_{0,i}^p, \alpha_{1,i}^p, \ldots, \alpha_{p,i}^p\right)$ the sequence of its mononial coefficients on $[i, i+1]$

$$\phi_p(t) = \alpha_{0,i}^p + \alpha_{1,i}^p t^2 + \ldots + \alpha_{p,i}^p t^p, \quad \forall t \in [i, i+1] \tag{11}$$

When $i, j \notin [0, p]$, we set $\alpha_{i,j}^p$

Theorem 2.9 (Taylor Coefficients)**.**

$$\alpha_{l,k}^p = \frac{k}{p}\alpha_{l,k}^{p-1} + \frac{1}{p}\alpha_{l-1,k}^{p-1} + \frac{p+1-k}{p}\alpha_{l,k-1}^{p-1} - \frac{1}{p}\alpha_{l-1,k-1}^{p-1} \tag{12}$$

Proof : Using the Cox-Deboor theorem (2.8), if $x = i + t$ where $t \in [0, 1]$, we have

$$\frac{x}{p}\phi^{p-1}(x) = \left(\frac{k}{p} + \frac{t}{p}\right)\left(\alpha_{0,i-1}^{p-1} + \alpha_{1,i-1}^{p-1}t^2 + \ldots + \alpha_{p,i-1}^{p-1}t^p\right)$$

$$\frac{p+1-x}{p}\phi^{p-1}(x-1) = \left(\frac{p+1-k}{p} - \frac{t}{p}\right)\left(\alpha_{0,i-1}^{p-1} + \alpha_{1,i-1}^{p-1}t^2 + \ldots + \alpha_{p,i-1}^{p-1}t^p\right)$$

Adding the last expressions together, we get the the expected relation for $\alpha_{p,i}^p$.

□

Theorem 2.10 (Inner product)**.**

$$\int_{\mathbb{R}} \phi_p^{(r)}(t)\phi_q^{(s)}(t+k)\,dt = (-1)^r \phi_{p+q+1}^{(r+s)}(p+1+k) = (-1)^s \phi_{p+q+1}^{(r+s)}(q+1-k) \tag{13}$$

2.2 Cardinal B-Splines evaluation using pp-form

Thanks to the theorem (2.9), we can compute analyticaly the Taylor coefficients for low B-Splines order (Tables 1, 2 and 3).

Interval	Taylor coefficients	
$[0,1]$	0 1	$\alpha_{0,1}$
$[1,2]$	1 −1	$\alpha_{1,1}$

Table 1: The linear Cardinal B-Spline Taylor coefficients.

Interval	Taylor coefficients			
$[0, 1]$	0	0	$\frac{1}{2}$	$\alpha_{0,2}$
$[1, 2]$	$\frac{1}{2}$	1	-1	$\alpha_{1,2}$
$[2, 3]$	$\frac{1}{2}$	-1	$\frac{1}{2}$	$\alpha_{2,2}$

Table 2: The quadratic Cardinal B-Spline Taylor coefficients.

Interval	Taylor coefficients				
$[0, 1]$	0	0	0	$\frac{1}{6}$	$\alpha_{0,3}$
$[1, 2]$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\alpha_{1,3}$
$[2, 3]$	$\frac{2}{3}$	0	-1	$\frac{1}{2}$	$\alpha_{2,3}$
$[3, 4]$	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{6}$	$\alpha_{1,3}$

Table 3: The cubic Cardinal B-Spline Taylor coefficients.

3 Cardinal B-Spline series

3.1 Scaled and translated Cardinal B-Splines

From now on, $h > 0$ will denote the mesh step. $h\mathbb{Z}$ is the uniform grid of width h. The scaled and translated Cardinal B-Spline of degree p is defined by

$$\phi_{i,h,p}(x) := \phi_p\left(\frac{x}{h} - i\right) \tag{14}$$

The support of $\phi_{i,h,p}$ is $[i, i + p + 1]h$. We introduce the sequence $(t_i)_{i\in\mathbb{Z}}$, where $t_i = ih, \ \forall i \in \mathbb{Z}$. The following result is another version of the Cox-Deboor theorem (2.8).

Theorem 3.1 (Cox-Deboor)**.** *$\forall t \in \mathbb{R}$ and $p \geq 1$, we have*

$$\phi_{i,h,p}(t) = \frac{t - t_i}{t_{i+p} - t_i}\phi_{i,h,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}}\phi_{i+1,h,p-1}(t) \tag{15}$$

with $\phi_{i,h,p}(t) = 0, \ \forall t \notin [t_i, t_{i+p+1}]$.

Proof : Using the definition of the scaled and translated Cardinal B-Spline and the theorem (2.8), we have

$$\phi_{i,h,p}(t) = \frac{t - ih}{hp}\phi_{i,h,p-1}(t) + \frac{h(p + 1 + i) - t}{hp}\phi_{i+1,h,p-1}(t) \tag{16}$$

The result is straightforward, using the fact that $t_{i+p} - t_i = t_{i+p+1} - t_{i+1} = hp$.

□

3.2 Cardinal Splines

Definition 3.1 (Cardinal Spline)**.** *A Cardinal Spline, or cardinal B-Spline serie, of degree p on the grid $h\mathbb{Z}$ is the linear combination*

$$\sum_{k\in\mathbb{Z}} c_k \phi_{k,h,p} \tag{17}$$

In figure (Fig. 2), we plot all non-vanishing Cardinal B-Splines on the interval $[2, 3]$. In

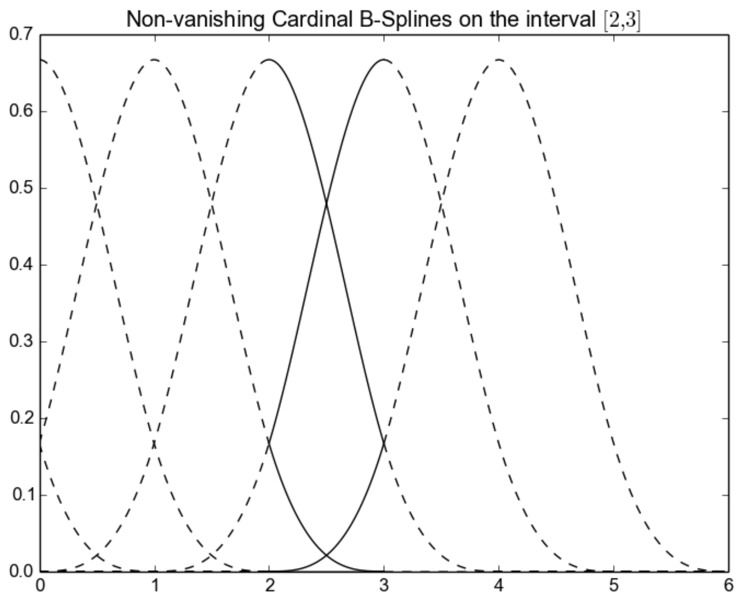

Figure 2: Non-vanishing Cardinal B-Splines on the interval $[2, 3]$

figure (Fig. 3), we plot a Cardinal B-Spline serie.

Proposition 3.2 (Marsden's identity)**.**

$$\forall x, t \in \mathbb{R}, (x-t)^p = \sum_{k\in\mathbb{Z}} m_{k,h,p}(t)\phi_{k,h,p}(x) \tag{18}$$

where $m_{k,h,p}(t) = h^p \Pi_{i=1}^{p}\left(k + i - \frac{t}{h}\right)$

Proposition 3.3 (Partition of unity)**.**

$$1 = \sum_{k\in\mathbb{Z}} \phi_{k,h,p}(t), \quad \forall t \in \mathbb{R} \tag{19}$$

Proposition 3.4 (Linear independence)**.** *For any element $[i, i+1]h$, the Cardinal B-Splines $\left(\phi_{k,h,p}\right)_{i-p\leq k\leq i}$ are linearly independent.*

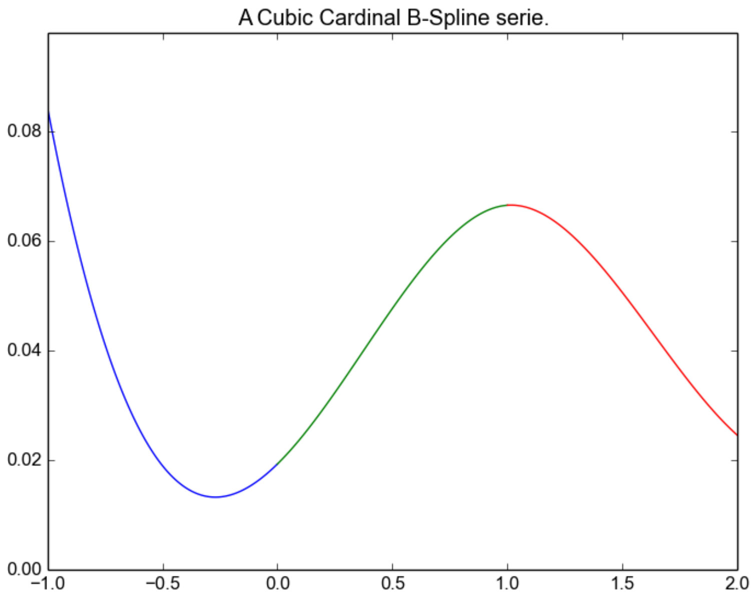

Figure 3: Example of a cubic Cardinal B-Spline serie.

4 Finite Elements using Cardinal B-Splines

In the context of the Finite Elements, we often deal with the following, mass, advection and stiffness matrices

$$M_{i_1 j_1} = \int_{\mathbb{R}} \phi_{i_1,h,p}(x)\, \phi_{j_1,h,p}(x)\, dx \tag{20}$$

$$A_{i_1 j_1} = \int_{\mathbb{R}} \dot{\phi}_{i_1,h,p}(x)\, \phi_{j_1,h,p}(x)\, dx \tag{21}$$

$$S_{i_1 j_1} = \int_{\mathbb{R}} \dot{\phi}_{i_1,h,p}(x)\, \dot{\phi}_{j_1,h,p}(x)\, dx \tag{22}$$

We assume periodic boundary conditions.

Remark 4.1. *In this case, these matrices are circulants, which is a special case of Toeplitz matrices.*

Using the change of variable $x \rightarrow \frac{x}{h} - (i_1 - j_1)$ and using the results of the last section, these matrices write

$$M_{i_1 j_1} = h\, \phi_{2p+1}(p + 1 - (i_1 - j_1)) \tag{23}$$

$$A_{i_1 j_1} = -\dot{\phi}_{2p+1}(p + 1 - (i_1 - j_1)) \tag{24}$$

$$S_{i_1 j_1} = h^{-1} \ddot{\phi}_{2p+1}(p + 1 - (i_1 - j_1)) \tag{25}$$

For better clarity we omit from now on, the spline degree p and the mesh step h from the definition of the scaled and translated cardinal B-Spline.
We also consider the matrices M_f, A_f, S_f where the value at the i_1^{th} line and j_1^{th} column are given, respectively, by

$$\left(M_f\right)_{i_1 j_1} = \int_{\mathbb{R}} f(x)\, \phi_{i_1,h,p}(x)\, \phi_{j_1,h,p}(x)\, dx \tag{26}$$

$$\left(A_f\right)_{i_1 j_1} = \int_{\mathbb{R}} f(x)\, \dot{\phi}_{i_1,h,p}(x)\, \phi_{j_1,h,p}(x)\, dx \tag{27}$$

$$\left(S_f\right)_{i_1 j_1} = \int_{\mathbb{R}} f(x)\, \dot{\phi}_{i_1,h,p}(x)\, \dot{\phi}_{j_1,h,p}(x)\, dx \tag{28}$$

5 Spline functions

The space of Cadinal Splines is a special case of a Schoenberg space. The regularity between two consecutive elements is equal to $p-1$ where p is the Cardinal B-Spline degree. We will denote it by $\mathcal{S}^p_{p-1}$. More generally, one can ask for any regularity $r_i,\ 0 \leq r_i \leq p-1$ between two elements e_i and e_{i+1}. In this case, the constructed space is known as the Schoenberg space and is denoted by $\mathcal{S}^p_{\mathbf{r}}$, where $\mathbf{r} = (r_1, \dots, r_{n-1})$ and n the number of subdivisions. When $r_i = p-1,\ 1 \leq i \leq n-1$, *i.e.* the maximal regularity, the Schoenberg space will be denoted by $\mathcal{S}^p$.

B-Splines using open knots vector : The drawback of the Cardinal B-Splines is the lack of interpolatory functions that allow us to impose, in a strong form, Dirichlet boundary conditions. We will show how to construt a Spline space where all basis functions vanish on the boundary.
Let's go back to the theorem (3.1). The idea is to use the same recurrence formula in order to create a new family, or sequence of B-Splines. Rather than having $t_i = ih,\ \ \forall i \in \mathbb{Z}$, we will take a finite sequence $(t_i)_{1\leq i\leq m}$, where $a = t_1 = t_2 = \ldots = t_{p+1}$ and $b = t_m = t_{m-1} = \ldots = t_{m-p}$. We assume also that the sequence $(t_i)_{p+1\leq i\leq m-p}$ is strictly increasing.
As we see, we can still use the recurrence formulae. Note that for the first and the last B-Splines, the formula is still valid as long as we consider that $\frac{0}{0} = 0$, whenever the length of the interval $[t_i, t_{i+p+1}]$ is zero. Moreover, for the B-Splines to be well defined, we need an initialization. As for the Cardinal B-Spline, we will assume that the first order (degree 0) is a constant equal to 1. The constructed sequence $(t_i)_{1\leq i\leq m}$ is known as an *open knot vector*. We usually write $m = n + p + 1$.
We note some important properties of a B-splines, denoted by N_i^p, basis:

- B-splines are piecewise polynomial of degree $p = k - 1$,
- When $n = k$, B-splines are exactly the Bernstein polynomials,
- Compact support; the support of N_j^k is contained in $\left[t_j, t_{j+k}\right]$,
- If $x \in\ ]t_j, t_{j+1}[$, then only the *B-splines* $\{N^k_{j-k+1}, \cdots, N^k_j\}$ are non vanishing at x,

- Positivity: $\forall j \in \{1, \cdots, n\} \;\; N_j(x) > 0, \;\; \forall x \in]t_j, t_{j+k}[$,
- Partition of unity : $\sum_{i=0}^{n-1} N_i^k(x) = 1, \forall x \in \mathbb{R}$,
- Local linear independence,
- $\forall i, \;\; p+1 \leq i \leq m-p$, the regularity of the B-Spline is $C^{(p-1)}$ at t_i
- Interpolation: $N_1(a) = 1$ and $N_n(b) = 1$.

The vectorial space spanned by these B-splines, which we denote $\mathcal{S}^p_{p-1}([a,b])$, is again called the Schoenberg space. The dimension of this space is n.
In figures (Fig. 4 and 5), we show the B-Splines generated using open knot vectors on a uniform grid.

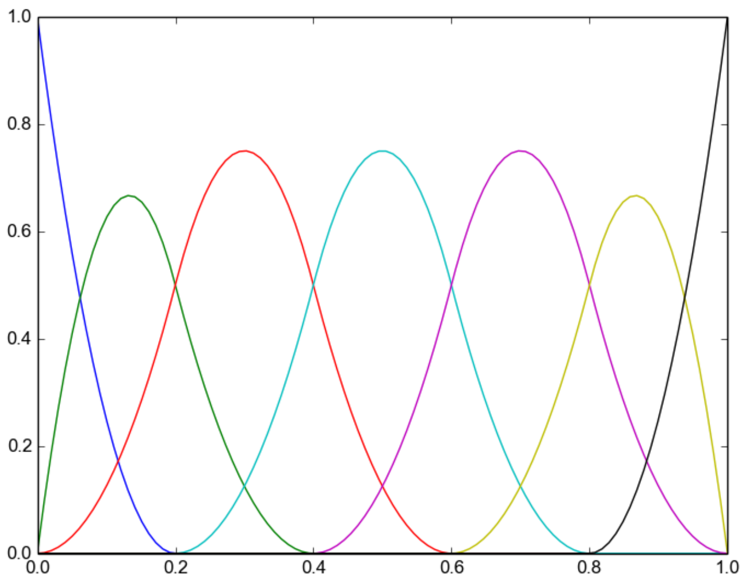

Figure 4: Quadratic B-Splines using the knot vector $\left\{0, 0, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, 1, 1\right\}$

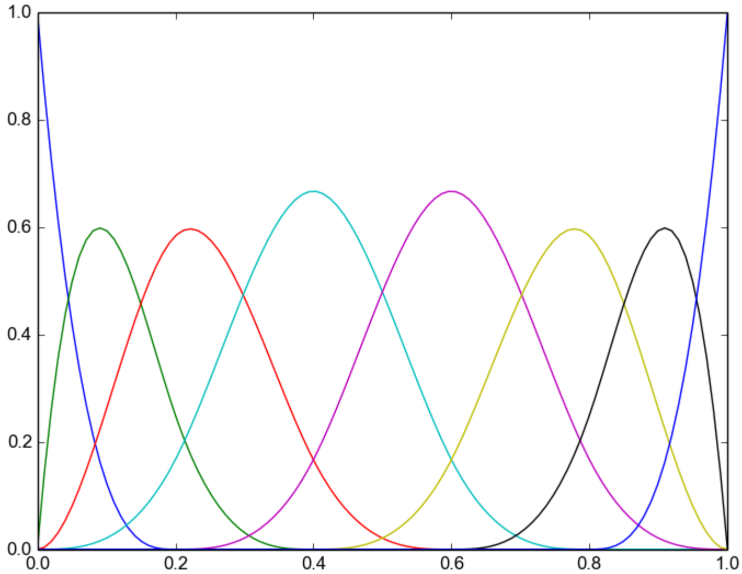

Figure 5: Cubic B-Splines using the knot vector $\left\{0, 0, 0, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, 1, 1, 1\right\}$

Exercises

Exercise 5.1 (Optional)**.** *Prove theorem 2.2*

Exercise 5.2 (Optional)**.** *Prove theorem 2.3*

Exercise 5.3 (Optional)**.** *Prove theorem 2.4*

Exercise 5.4 (Optional)**.** *Prove theorem 2.5*

Exercise 5.5 (Optional)**.** *Prove theorem 2.6*

Exercise 5.6 (Optional)**.** *Prove theorem 2.7*

Exercise 5.7. *Finite Element for the 1D Laplace problem:*

$$-u'' + \mu u = f \text{ in } [0,1] \quad \mu \geq 0 \tag{29}$$

1. *Write the variational formulation for periodic boundary conditions $u(0) = u(1)$, and prove that this variational formulation admits a unique solution, up to a constant. (Hint: add a constraint like $\int_0^1 u(x)dx = 0$)*
2. *Let us consider a uniform grid of width h of the interval $[0,1]$. We define the space of Cardinal Splines $\mathcal{S}_{h,p} = \{\sum_{k\in\mathbb{Z}} c_k \phi_{k,h,p}, \quad (c_k)_{k\in\mathbb{Z}}\}$.*
   - *(a) show that using a convenient periodic indexation, the space $\mathcal{S}_{h,p}$ can be used an approximation space. (Hint: find N, such that you can repalce $\mathbb{Z}$ by the cyclic group $\mathbb{Z}/N\mathbb{Z}$)*
   - *(b) Express the different matrices involved in the Finite Element approximation with respect to integrals over the basis functions and the data.*
   - *(c) Express the coefficients of the global matrices with respect to the Cardinal B-Spline of higher degree.*
3. *Show that the constructued matrices are circulants and express their eigenvalues and eigenvectors.*
4. *Using the Fast Fourier Transform, give an algorithm solving the obtained linear system.*

Exercise 5.8. *Finite Element for the 2D Laplace problem:*

$$-\nabla^2 u + \mu u = f \text{ in } [0,1]^2 \quad \mu \geq 0 \tag{30}$$

1. *Write the variational formulation for periodic boundary conditions, and prove that this variational formulation admits a unique solution, up to a constant. (Hint: add a constraint like $\int_{[0,1]^2} u(x,y)dxdy = 0$)*
2. *Let us consider a uniform grid of width h of the interval $[0,1]$. We define the space of Cardinal Splines $\mathcal{S}_{h,p} = \{\sum_{k\in\mathbb{Z}} c_k \phi_{k,h,p}, \quad (c_k)_{k\in\mathbb{Z}}\}$.*

(a) *show that using a convenient periodic indexation, the space $\mathcal{S}_{h,p} \times \mathcal{S}_{h,p}$ can be used an approximation space. (Hint: find N, such that you can repalce $\mathbb{Z}$ by the cyclic group $\mathbb{Z}/N\mathbb{Z}$)*

(b) *Express the different matrices involved in the Finite Element approximation with respect to integrals over the basis functions and the data.*

(c) *Express the coefficients of the global matrices using the $1D$ matrices of the previous exercise and the Kronecker product.*

3. *Show that solving the obtained linear system is equivalent to following problem*
   Find $X \in \mathbb{R}^N \times \mathbb{R}^N$, such that

$$SXM + MXS = F \tag{31}$$

   where the vector of unknowns (and right hand side) are viewed as matrices in $X \in \mathbb{R}^N \times \mathbb{R}^N$.

4. *Using the Fast Fourier Transform, give an algorithm solving the linear system.*

From now on, we only consider Schoenber spaces spanned with B-Splines, constructed using open knot vectors.

Exercise 5.9. *Show how to construct a finite dimentional subspace of $H^1_0(\Omega)$, with open knot vectors, where Ω is a $1D$ and $2D$ domain.*

Exercise 5.10. *Show that the space $\left\{\frac{d}{dt}u : u \in \mathcal{S}^p\right\}$ is equal to $\mathcal{S}^{p-1}$.*

Exercise 5.11 (Optional)**.** *Let $\Omega \subset \mathbb{R}^2$. For every $\phi \in H^1(\Omega)$, we have $\nabla \times \nabla\phi = 0$, therefor $\nabla\phi \in H(curl, \Omega)$. When Ω is simply connected, we know that* $\mathrm{Im}(\nabla) = \mathrm{Ker}(\nabla\times)$. *On the other hand, we have $\nabla \cdot \nabla \times \phi = 0$ and $\nabla \times \phi \in Hdiv$. We also have, for simply connected domains,* $\mathrm{Im}(\nabla\times) = \mathrm{Ker}(\nabla\cdot)$. *All these results are summerized in the following De Rham diagrams:*

$$H^1(\Omega) \xrightarrow{\boldsymbol{\nabla}} H(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} L^2(\Omega) \tag{32}$$

$$H^1(\Omega) \xrightarrow{\boldsymbol{\nabla}\times} H(\mathrm{div}, \Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \tag{33}$$

We assume that the computational domain is the unit square. We can construct a Spline space using the tensor product. We introduce the space $\mathcal{S}^{p,q} := \mathcal{S}^p \times \mathcal{S}^q$.
The aim of this exercise, is to show that the following discrete diagrams hold also.

$$\begin{array}{ccccc} H^1(\Omega) & \xrightarrow{\boldsymbol{\nabla}} & H(\mathrm{curl}, \Omega) & \xrightarrow{\nabla\times} & L^2(\Omega) \\ \cup & & \cup & & \cup \\ \mathcal{S}^{p,p} & \longrightarrow & \mathcal{S}^{p-1,p} \times \mathcal{S}^{p,p-1} & \longrightarrow & \mathcal{S}^{p-1,p-1} \end{array} \tag{34}$$

$$
\begin{array}{ccccc}
 & \nabla\times & & \nabla\cdot & \\
H^1(\Omega) & \longrightarrow & H(\mathrm{div},\Omega) & \longrightarrow & L^2(\Omega) \\
\cup & & \cup & & \cup \\
\mathcal{S}^{p,p} & \longrightarrow & \mathcal{S}^{p,p-1}\times\mathcal{S}^{p-1,p} & \longrightarrow & \mathcal{S}^{p-1,p-1}
\end{array}
\tag{35}
$$

1. *Let us consider the first discrete diagram,*

   (a) *If $\phi \in \mathcal{S}^{p,p}$, show that $\nabla\phi \in \mathcal{S}^{p-1,p} \times \mathcal{S}^{p,p-1}$.*

   (b) *If $\mathbf{\Psi} \in \mathcal{S}^{p-1,p} \times \mathcal{S}^{p,p-1}$, show that $\nabla \times \mathbf{\Psi} = 0$ and $\nabla\left(\mathcal{S}^{p,p}\right) \subset \mathrm{Ker}(\nabla\times) \bigcap \mathcal{S}^{p-1,p} \times \mathcal{S}^{p,p-1}$.*

   (c) *Show that the last inclusion, is an equality and that the first discrete diagram holds. (Hint: prove that both spaces have the same dimension and that the operator $\nabla\times$ is surjective.)*

2. *Let us consider the second discrete diagram,*

   (a) *If $\phi \in \mathcal{S}^{p,p}$, show that $\nabla \times \phi \in \mathcal{S}^{p,p-1} \times \mathcal{S}^{p-1,p}$.*

   (b) *If $\mathbf{\Psi} \in \mathcal{S}^{p,p-1}\times\mathcal{S}^{p-1,p}$, show that $\nabla\cdot\mathbf{\Psi} = 0$ and $\nabla\times(\mathcal{S}^{p,p}) \subset \mathrm{Ker}(\nabla\cdot) \bigcap \mathcal{S}^{p,p-1}\times\mathcal{S}^{p-1,p}$.*

   (c) *Show that the last inclusion, is an equality and that the first discrete diagram holds. (Hint: prove that both spaces have the same dimension and that the operator $\nabla\cdot$ is surjective.)*

Exercise 5.12. *Using the results of the last exercise, write the weak formulation, the corresponding discrete system and the involved matrices for the Time Harmonic Maxwell problem.*

Exercise 5.13.

1. *write the weak formulation, the corresponding discrete system and the involved matrices for the (TE mode) Time Domain Maxwell problem using the second De Rham sequence.*

2. *For the basis of $\mathcal{S}^{p-1,p} \times \mathcal{S}^{p,p-1}$, we choose to scale directions of lower order, i.e.*

$$
\boldsymbol{\psi}^1_{i,j} = \begin{pmatrix} N_i^p(x) D_j^{p-1}(y) \\ 0 \end{pmatrix}, \quad \boldsymbol{\psi}^2_{i,j} = \begin{pmatrix} 0 \\ D_i^{p-1}(x) N_j^p(y) \end{pmatrix}
$$

   we have

$$
W_{div} = \mathrm{span}\left\{\boldsymbol{\psi}^1_{i,j}, \boldsymbol{\psi}^2_{i,j}, \quad 1 \le i \le N_x,\ 1 \le j \le N_y\}\right\},
$$

   where $D_i^p = \frac{p}{t_{i+p}-t_i} N_i^{p-1}(t)$

3. *If we denote by M_E the mass matrix on the space $\mathcal{S}^{p,p-1} \times \mathcal{S}^{p-1,p}$, M_B the mass matrix on the space $\mathcal{S}^{p,p}$ and K the matrix related to the operator $\int_\Omega H \nabla \times \mathbf{\Psi}$, show that the matrix $M_E^{-1}K^T$ is the matrix of incidence (consisting of two blocks of the discrete derivatives in the x and y directions). (Hint: take the curl of a function in $\mathcal{S}^{p,p}$ and use the fact that $N_i^{p\prime}(t) = D_i^p(t) - D_{i+1}^p(t)$)*

4. *Show that the linear system to solve can be written as:*

$$\left\{ \begin{array}{c} \dot{\mathbf{E}} = RH \\ M_B\dot{H} = -K^T\mathbf{E} \end{array} \right. \tag{36}$$

5. *We use a Leap-Frog time discretization scheme. We recall the general formulation of the LF scheme. For the linear system*

$$\left\{ \begin{array}{c} M_W\dot{\mathbf{e}} = Kh \\ M_V\dot{h} = -K^T\mathbf{e} \end{array} \right.$$

the LF time discretization of order N is given by

$$\left\{ \begin{array}{c} M_W\frac{E^{n+1}-E^n}{\Delta t} = K_N H^{n+\frac{1}{2}} \\ M_V\frac{H^{n+\frac{3}{2}}-H^{n+\frac{1}{2}}}{\Delta t} = -K_N^T E^{n+1} \end{array} \right.$$

where

$$\left\{ \begin{array}{cc} K_N = K & ,N=2 \\ K_N = K(I - \frac{\Delta t^2}{24}M_W^{-1}KM_V^{-1}K^T) & ,N=4 \end{array} \right.$$

We define the global energy by $\mathcal{E}^n := \frac{1}{2}\{(E^n)^T M_W E^n + (H^{n-\frac{1}{2}})^T M_V H^{n+\frac{1}{2}}\}$
Show that the global energy is stationary, i.e $\mathcal{E}^{n+1} = \mathcal{E}^n$

6. *The stability of the scheme depends on the global energy, which must be a positive quadratic form to ensure stability.*
Show that $\mathcal{E}^n$ *is a positive quadratic form if* $\Delta t \leqslant \frac{2}{d_N}$*, where* $d_N = \|M_V^{-\frac{1}{2}}K_N^T M_W^{-\frac{1}{2}}\|$