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Advanced finite element methods for MHD, WS 2016

http://www-m16.ma.tum.de/Allgemeines/AdvFE16

Exercise sheet 1 (November 28, 2016)

1D elliptic problem with Galerkin FEM

Write a Galerkin finite element (FEM) solver for the elliptic equation

$$\begin{cases} -u''(x) + u(x) = f(x)\,, & x \in (a,b) \subset \mathbb{R}\,, \\ u(a) = u(b) = 0\,. \end{cases} \tag{1}$$

for given $f \in C(\overline{\Omega})$ with $\Omega = (a,b)$. We search for numerical solutions $u_h \in V_h$, where $V_h \subset V$ is a subspace of $V = H^1_0(\Omega)$. For a Galerkin method, three main tasks need to be performed: triangulation of the domain, writing a basis of the subspace V_h, and solving the linear system arising from the weak formulation of (1) in the space V_h.

1. Triangulation: we compute a partition of the domain $[a,b]$ into M *elements*, where the vertices (interfaces between elements) are denoted by $a = x_0, x_1, \dots, x_M = b$. The i-th element is denoted by $K_i = [x_i, x_{i+1}]$, $i = 0, \dots, M-1$, its volume is $\mathcal{C}_i = x_{i+1} - x_i$. A non-uniform partition can be created with the help of a positive *spacing function* $\mathcal{H} : [a,b] \to \mathbb{R}^+$. The vertex x_i is the solution of

$$c \int_a^{x_i} \frac{1}{\mathcal{H}(x)}\,dx = i\,, \qquad \text{with} \quad c = \frac{M}{\int_a^b 1/\mathcal{H}(x)\,dx}\,. \tag{2}$$

   Setting $g'(x) = c/\mathcal{H}(x)$, $g(a) = 0$, the x_i are the roots of $f_i := g(x) - i$. A second more convenient strategy is to compute preliminary vertices $\tilde{x}_{i+1} = \tilde{x}_i + \mathcal{H}(\tilde{x}_i)$ starting from $\tilde{x}_0 = a$, until $i + 1 = P$ with $\tilde{x}_P > b$. Then $M = P$ and we repeat the process as

$$x_0 = a\,, \qquad x_{i+1} = x_i + c\,\mathcal{H}(x_i)\,, \qquad \text{with} \quad c = \frac{b-a}{\tilde{x}_P - a}\,. \tag{3}$$

2. Basis functions: recall that $H^1(\Omega) \subset C^0(\overline{\Omega})$, such that the approximation space $V_h \subset V$ consists of functions that are continuous on (a,b). We choose $V_h = X_h^r$, where

$$X_h^r = \{v_h \in C^0([a,b]) : \, v_h|_{K_i} \in \mathbb{P}_r \ \forall\, i,\ v_h(a) = v_h(b) = 0\}\,, \tag{4}$$

   the space of piece-wise polynomials of degree r, vanishing at the boundary. Construct bases (φ_i) for X_h^1 and X_h^2 with Lagrange polynomials and with splines, and for X_h^3 only with splines (a short introduction of how to use B-splines in Matlab can be found in last semester's course CompPlasmaPhys16, Exercise 5). Map the basis functions to the *reference element* $[0,1]$ via

$$R_i : [0,1] \to K_i\,, \quad \xi \mapsto x\,, \quad x = R_i(\xi) = x_i + \xi(x_{i+1} - x_i)\,. \tag{5}$$

3. Linear solve: write the weak formulation of problem (1) in the space V_h, by expanding the trial and the test function as

$$u_h(x) = \sum_i u_i \, \varphi_i(x) \,, \qquad v_h = \sum_i v_i \, \varphi_i(x) \,. \tag{6}$$

Compute the *stiffness matrix* $S = (s_{ij})$, the *mass matrix* $M = (m_{ij})$ and the right-hand side $b = (f_i)$,

$$s_{ij} = \int_a^b \varphi_i'(x)\varphi_j'(x)\,dx \,, \qquad m_{ij} = \int_a^b \varphi_i(x)\varphi_j(x)\,dx \,, \qquad f_i = \int_a^b f(x)\varphi_i(x)\,dx \,. \tag{7}$$

where $i, j = 1, \ldots, \dim X_h^r$. For $r \leq 2$, compute (s_{ij}) and (m_{ij}) analytically on the reference element. Save the matrices using the `sparse` command in Matlab. Then use a direct method to solve the linear system $(S + M)\hat{u} = b$ for $\hat{u} = (u_i)$.

Test the convergence rate w.r.t $h = \max(\mathcal{C}_i)$ on uniform ($\mathcal{H} = const.$) and non-uniform grids, in the L^2- and in the H^1-norm. As the reference use polynomials of the form $(x - a)(x - b)p(x)$, with polynomials $p(x)$ of degree 2,4,6 and 8. As a non-polynomial reference use the manufactured solution

$$u_{ex}(x) = \sin(2\pi x/(b - a))\, e^x \,. \tag{8}$$

Plot the results. As a quadrature rule for integrals, you could use e.g. Boole's rule:

$$\int_{y_1}^{y_5} f(x)\,dx = \frac{h}{90}\Big[7f(y_1) + 32f(y_2) + 12f(y_3) + 32f(y_4) + 7f(y_5)\Big] + \mathcal{O}(h^7)\,, \tag{9}$$

where $h = (y_5 - y_1)$ and $y_1, \ldots, y_5$ are five equally spaced points in the interval $[y_1, y_5]$. You can also try more or less accurate quadrature rules and see the influence on the error computation.

4. Bonus: construct new bases for X_h^2 on the reference element, $\xi \in [0, 1]$, using the *shape functions*

   (a) $\hat{\varphi}_0 = (1 - \xi)\,, \qquad \hat{\varphi}_1 = \xi\,, \qquad \hat{\varphi}_2 = (1 - \xi)\xi \quad,$

   (b) $\hat{\varphi}_0 = 1\,, \qquad \hat{\varphi}_1 = \xi\,, \qquad \hat{\varphi}_2 = \xi^2 \quad.$

   Recall that continuity needs to be enforced at the vertices (x_i)!