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Advanced finite element methods for MHD, WS 2016

http://www-m16.ma.tum.de/Allgemeines/AdvFE16

Exercise sheet 2 (January 23, 2017)

Mixed FEM for linearized MHD in 2D

As shown in the supplementary notes, the fully-implicit time discretization of the linearized, two-dimensional, ideal MHD equations leads to equations of the form

$$\begin{cases} \mathbf{u} + \dfrac{B_0}{\mu_0}\nabla B = \mathbf{f}\,, \\ B + B_0 \nabla\cdot\mathbf{u} = g\,, \end{cases} \tag{1}$$

for the unknowns $\mathbf{u}(x,y) \in \mathbb{R}^2$, the plasma velocity in the poloidal plane, and $B(x,y) \in \mathbb{R}$, the magnetic field perturbation along the equilibrium magnetic field, with given data $\mathbf{f}(x,y) \in \mathbb{R}^2$, $g(x,y) \in \mathbb{R}$ for $(x,y) \in \Omega = [0,L_x] \times [0,L_y]$. Inserting $\mathbf{u}$ from the first into the second equation yields

$$B - \frac{B_0^2}{\mu_0}\nabla\cdot\nabla B = h\,, \qquad h := g - B_0\nabla\cdot\mathbf{f}\,, \tag{2}$$

which corrrsponds to an implicit time discretization of a wave equation for B, with wave speed $v_A = B_0/\sqrt{\mu_0}$, the famous *Alfvén velocity* (for unit density). In what follows we supplement (1) and (2) with the natural boundary conditions $B = 0$ on $\partial\Omega$ and assume $g, h \in L^2(\Omega)$.

1. Write down two different weak formulations of the *mixed problem* (1) and one weak formulation of the *primordial problem* (2).
2. Try to apply the Lax-Milgram Lemma for existence and uniqueness of solutions.
3. Assuming existence and uniqueness of solutions, show that the solutions of your weak formulations of (1) and (2) are equivalent.
4. Assume a triangulation $\mathcal{T}_h$ of the domain Ω consisting of rectangles K_i, $i = 1,\dots,N_{el}$. For our problem, the Raviart-Thomas space of order k is defined as

   $$RT^k := \left\{ \mathbf{v} = (v_x, v_y) \in H(\mathrm{div},\Omega) : v_x|_{K_i} \in \mathbb{Q}_{k+1,k} \text{ and } v_y|_{K_i} \in \mathbb{Q}_{k,k+1}\ \forall\, i \right\}.$$

   For $\mathbf{v} \in RT^k$ one has $\mathrm{div}(\mathbf{v})|_{K_i} \in \mathbb{Q}_{k,k}$. Construct Lagrange bases for RT^0, RT^1 and RT^2 on the reference element $\hat{K} = [0,1] \times [0,1]$. To define the degrees of freedom, use Gauss-Lobatto points when continuity needs to be enforced (shared degrees of freedoms on the interface) and Gauss-Legendre points otherwise. The Gauss points can be computed from the scripts 'lglnodes.m' and 'lgwt.m', respectively, provided on the courses' webpage. Plot the basis functions in Matlab.
5. Compute the block matrices arising from the FEM discretization of system (1) with Lagrange basis functions from Gauss points. Show that the mass matrices are diagonal.