1 Introduction

2 Cardinal B-Splines

Cardinal B-Splines play an important role in the approximation theory (multi-resolution approximation, ...). In the sequel, we shall give a definition of the Cardinal B-Spline using the convolution operator. Then, we will present some of the most important properties, at least needed when using uniform B-Splines in a Finite Elements method.

Definition 2.1. A cardinal B-spline of zero degree, denoted by $\phi_0$, is the characteristic function over the interval $[0, 1)$, i.e.,

$$\phi_0(t) := \begin{cases} 1, & t \in [0, 1) \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (1)

A cardinal B-Spline of degree $p$, $p \in \mathbb{N}$, denoted by $\phi_p$, is defined by convolution as

$$\phi_p(t) = (\phi_{p-1} * \phi_0)(t) = \int_{\mathbb{R}} \phi_{p-1}(t-s)\phi_0(s) \, ds$$  \hspace{1cm} (2)

Example 1: When $p = 1$, it is easy to show that

$$\phi_1(t) := \begin{cases} t, & t \in [0, 1) \\ 2-t, & t \in [1, 2) \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (3)

When $p = 2$, it is easy to show that

$$\phi_2(t) := \begin{cases} \frac{1}{2}x^2, & t \in [0, 1) \\ \frac{1}{2} + (x-1) - (x-1)^2, & t \in [1, 2) \\ \frac{1}{2} - (x-2) + \frac{1}{2}(x-2)^2, & t \in [2, 3) \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (4)

In figure (Fig. 1), we plot the Cardinal B-Splines of degrees 1, 2, 3 and 4.

Remark 2.1. The colored area under the graph of $\phi_2$ represents the average $\int_{t_1}^{t_2} \phi_2(t) \, dt$ which is the value $\phi_3(x)$. 

In figure (Fig. 1), we plot the Cardinal B-Splines of degrees 1, 2, 3 and 4.
Figure 1: Cardinal B-Splines of degrees 1, 2, 3 and 4
2.1 Cardinal B-Splines properties

Let \( \phi_p \) be a cardinal B-Spline of degree \( p, p \in \mathbb{N}. \) The following properties can be proved by induction on the B-Spline degree \( p. \)

**Theorem 2.2** (Minimal support). The support of \( \phi_p \) is \([0, p + 1]\)

**Theorem 2.3** (Positivity). \( \phi_p(s) \geq 0, \forall s \in [0, p + 1] \)

**Theorem 2.4.** \( \phi_p \in C^{p-1} \)

**Theorem 2.5.** \( \phi_p \) is a piecewise-polynomial of degree \( p \) at each interval \([i, i+1], \forall i \in \{0, 1, \ldots, p\}\)

The sequence \( \{0, 1, 2, \ldots, p\} \) is known as the **breaks** of the cardinal B-Spline of degree \( p. \)

**Theorem 2.6.** \( \forall t \in [0, p + 1] \) and \( p \geq 1, \) we have

\[
\dot{\phi}_p(t) = \phi_{p-1}(t) - \phi_{p-1}(t - 1)
\]  

(5)

**Theorem 2.7** (Symmetry). \( \phi_p \) is symmetric on the interval \([0, p + 1], \) i.e.

\[
\phi_p(t) = \phi_p(p + 1 - t), \quad \forall t \in [0, p + 1]
\]  

(6)

The following theorem was proved in 1972 by both Cox and Deboor separately.

**Theorem 2.8** (Cox-Deboor). \( \forall t \in [0, p + 1] \) and \( p \geq 1, \) we have

\[
\phi_p(t) = \frac{t}{p} \phi_{p-1}(t) + \frac{p + 1 - t}{p} \phi_{p-1}(t - 1)
\]  

(7)

**Proof:** Let \( \phi'_p(t) := \phi_p(t - i), \forall t \in [0, p + 1]. \) We will proof the result by induction. Since both sides vanish at \( t = 0, \) we will use the equivalence to the formula for the derivative 2.6.

\[
\phi^{p-1}_0 - \phi^{p-1}_1 = \frac{1}{p} \left( \phi^{p-1}_0 - \phi^{p-1}_1 \right) + \left[ \frac{t}{p} \left( \phi^{p-2}_0 - \phi^{p-2}_1 \right) + \frac{p + 1 - t}{p} \left( \phi^{p-2}_1 - \phi^{p-2}_2 \right) \right]
\]  

(8)

The last term of the previous relation, can be written as

\[
\frac{p - 1}{p} \left[ \left( \frac{t}{p - 1} \phi^{p-2}_0 + \frac{p - t}{p - 1} \phi^{p-2}_1 \right) - \left( \frac{t - 1}{p - 1} \phi^{p-2}_1 + \frac{p - (t - 1)}{p - 1} \phi^{p-2}_2 \right) \right]
\]  

(9)

Now, if we assume that the recursion is valid up to \( p - 1, \) then the last terms is equal to

\[
\frac{p - 1}{p} \left( \phi^{p-1}_0 - \phi^{p-1}_1 \right)
\]  

(10)
Appendix to B-splines:

1) A cardinal B-spline of order \( p \) is defined by a knot vector
\[ K_p = [u_0, u_1, \ldots, u_{p+1}] \]

The spline is \( p-1 \) regular at the knots, \( \phi_p \in C^{p-1}(K_p, \mathbb{R}) \).
If a knot appears \( m \leq p+1 \) times, the regularity at that knot is reduced by \( m-1 \).

2) Use the Whitney function, "bsplignu" to explore the properties of B-splines.

3) A spline basis of \( X_n^p \) on the interval \([a, b]\) is constructed as follows:

1) Partition into \( m \) subintervals,
\[ \tau = [a-x_0, x_1, x_2, \ldots, x_{m-1}, x_m = b] \]

2) Periodic boundary conditions:

Ex. \( (p=3) \):
\[ \tau = [a, x_1, x_2, x_3, x_4, x_5, \ldots, b] \]

\[ K_3^{(1)} \]

\[ K_3^{(2)} \]

\[ K_3^{(3)} \]

\[ \Rightarrow \text{dim } X_n^p = m-p \]

Transform each knot vector to \([6,4,2,3,4]\).
3) Homogeneous Dirichlet boundary conditions:
   
   i) Construct a new knot vector $\mathbf{T}'$, where the first and the last knot appear $p+1$ times,
   
   $\mathbf{T}' = \{a, a, \ldots, a, x_1, x_2, \ldots, x_n, a, \ldots, b, b, b, \ldots, b\}$
   
   $p+1$ times
   
   $p+1$ times
   
   ii) Remove the first and the last (basis) spline functions, created from
   
   $[a, \ldots, a, x_i, \ldots, x_n, b, \ldots, b]$. The remaining functions are the desired basis.