1D Poisson solver with finite differences

The aim is to numerically solve the Poisson equation

$$-\phi''(x) = \rho(x), \quad x \in (a, b) \subset \mathbb{R},$$  \hspace{1cm} (1)

for given $\rho$, continuous on the interval $(a, b)$, and suitable boundary conditions. We use finite differences to approximate the equation. The domain $[a, b]$ (for generality with boundary points) is divided into $N$ equally large cells, yielding $N + 1$ grid points $x_i \in [a, b], i = 0, \ldots, N$, on which the discrete solution is defined.

1. **Dirichlet boundary conditions.** The values of $\phi$ are given at the boundary:

$$\phi(a) = \alpha, \quad \phi(b) = \beta.$$  \hspace{1cm} (2)

(a) Show that the problem (1),(2) has a unique solution. (Hint: suppose $\phi$ and $\psi$ are both solutions of (1), then formulate the Laplace equation for $\eta := \phi - \psi$ (with boundary conditions). Multiply this equation by $\eta$ and integrate over $[a, b]$).

(b) Write a finite difference solver for (1),(2) for arbitrary $a, b, \alpha, \beta, N$ and $\rho$ by approximating the Laplacian at the grid points $i = 1, \ldots, N - 1$ via

$$\phi''(x_i) \approx \frac{1}{h^2} \left( \phi_{i+1} - 2\phi_i + \phi_{i-1} \right).$$  \hspace{1cm} (3)

where $\phi_i = \phi(x_i)$ and $h$ is the cell size (grid spacing). For given $N$, check the eigenvalues of the system matrix using the MATLAB command `eig`.

(c) Set $a = 0, b = 2\pi, \rho(x) = 2\sin(x) + x\cos(x), \alpha = 0, \beta = 2\pi$ and solve (1),(2) for $N = 8, 16, 32, 64, 128, 256$. In each run, compute and save the errors with respect to the true solution $\phi(x) = x\cos(x)$ in the $L^1$-, the $L^2$- and the $L^\infty$-norm. Plot the numerical solution along with the true solution using the command `plot`.

(d) Convergence tests: once you completed all the runs, plot the errors as a function of $h$. Does the solution converge as $h \to 0$? If so, at the expected rate? Can you compute the expected convergence rate from (3)? (Hint: insert Taylor expansion).

2. **Mixed boundary conditions.** The derivative of $\phi$ is now given at $x = b$:

$$\phi(a) = \alpha, \quad \phi'(b) = \gamma.$$  \hspace{1cm} (4)

There are now $N$ unknowns to be determined, since $\phi(b)$ is not given. Does the problem (1),(4) have a unique solution (see 1.a)?
(a) Approximate the derivative at the boundary via \( (\phi_{N+1} - \phi_{N-1})/(2h) = \gamma \) and implement a finite difference solver for (1),(4). Check the eigenvalues of the system matrix. Launch your solver with the same parameters as in 1.c (except for the \( \beta \)-value) and with \( \gamma = 1 \). Perform convergence tests as in 1.d and plot the results.

(b) Approximate the derivative at the boundary via \( (\phi_{N+1} - \phi_{N})/h = \gamma \) and repeat the same tasks as in 2.a. Is there a difference in the convergence rate? If so, explain why.

3. **Periodic boundary conditions.** The solution is now defined on \( \mathbb{R} \), assumed in \( C^\infty \) and periodic with period \( L = b - a \):

\[
\phi(x + L) = \phi(x), \quad \forall x \in \mathbb{R}.
\]  

(5)

It is sufficient to compute the solution in \( [a, b) \) to know it on whole \( \mathbb{R} \). There are now \( N \) unknowns to be determined, since \( \phi(b) = \phi(a) \) and thus only one boundary point has to be computed.

(a) Is there a solvability condition on \( \rho \)? Does the problem (1),(5) have a unique solution (see 1.a)?

(b) Discretize the problem (1), (5) assuming \( \phi_N = \phi_0 \) and \( \phi_{-1} = \phi_{N-1} \). Check the eigenvalues of the system matrix \( A \). Is it invertible?

(c) Render the problem well-posed by imposing \( \phi(0) = 0 \), which leads to a modified system matrix \( A' \). Check the eigenvalues of \( A' \).

(d) Set \( a = 0, \ b = 2\pi \) and \( \rho = 4 \sin(2x) \), launch your solver for different values of \( N \) and perform the usual convergence tests. The true solution is obviously \( \phi(x) = \sin(2x) \). Plot the results.