1. **Constant-coefficient advection.**

For \( x \in [0, L] \) and \( t \in [0, T] \), consider the advection problem

\[
\begin{cases}
\frac{\partial}{\partial t} u(t, x) + a \frac{\partial}{\partial x} u(t, x) = 0, \quad a \in \mathbb{R}, \\
u(0, x) = u_0(x) = \frac{1}{(2\pi \sigma^2)^{1/2}} e^{-\frac{(x-L/2)^2}{2\sigma^2}},
\end{cases}
\]

(1)

where \( u \) is \( L \)-periodic. Denoting by \( u^n_j := u(t_n, x_j) \) on a uniform space-time grid with parameters \( \Delta t \) and \( h \), implement the following solvers for problem (1), for arbitrary values of \( L, T, a \) and \( \sigma \):

(a) an explicit Euler upwind scheme,

\[
u_{j}^{n+1} = \frac{\Delta t}{h} \left[ a \left( u_{j+1}^n - u_j^n \right) + a^+ \left( u_j^n - u_{j-1}^n \right) \right], \quad a^- = \min(0, a), \quad a^+ = \max(0, a).
\]

(b) a Lax-Wendroff scheme,

\[
u_{j}^{n+1} = \frac{\Delta t}{2h} \left( u_{j+1}^n - u_{j-1}^n \right) + \frac{a^2 \Delta t^2}{4h^2} \left( u_{j+1}^n - 2u_j^n + u_{j-1}^n \right).
\]

(c) a spectral scheme (solve analytically in Fourier space),

\[
\hat{u}_k^{n+1} = \exp \left( -\frac{2\pi i}{L} k a \Delta t \right) \hat{u}_k^n, \quad k = -N/2, \ldots, N/2 - 1,
\]

where \( \hat{u}_k = P_k^* u_j \) is the \( k \)-th discrete Fourier mode.

Solve (1) with these schemes for \( L = 1, T = 1.5, a = \pm 2 \) and \( \sigma = 0.05 \). Check the conservation of total mass, maximum of \( u \) and of the \( L^2 \)-norm. Make sure the CFL condition is satisfied for stability in the schemes a) and b). Use different spatial discretisations in order to perform convergence tests, by comparing to the exact solution \( u_e(t, x) = u_0(x - at) \).

Visualise your results and make a video of the evolution of \( u \).

2. **1D-1V Vlasov-Poisson.**

We shall apply our knowledge gained in the previous exercises for solving the Vlasov-Poisson system for electrons,

\[
\begin{aligned}
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial v} &= 0, \\
- \frac{\partial^2 \phi}{\partial x^2} &= 1 - \int f dv, \quad E = -\frac{\partial \phi}{\partial x},
\end{aligned}
\]

(2)
Here, $f > 0$ is a function of $t \in [0, T]$, $x \in [0, L_x]$ and $v \in [v_{\text{min}}, v_{\text{max}}]$, $v_{\text{min}} < 0 < v_{\text{max}}$, assumed $L_x$-periodic and $(v_{\text{max}} - v_{\text{min}})$-periodic, respectively, and $\phi = \phi(t, x)$ is $L_x$-periodic.

(a) Using a (periodic) Poisson solver, implement a solver for (2) via a splitting method for the Vlasov equation, i.e. at each time step solve sequentially the two problems $(A)$ and $(B)$, for $t \in [t_n, t_{n+1}]$,

\begin{align*}
(A) : \quad & \frac{\partial f^*}{\partial t} - E \frac{\partial f^*}{\partial v} = 0, \\
(B) : \quad & \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0,
\end{align*}

$f^*(t_n) = f(t_n)$, \quad $f(t_n) = f^*(t_{n+1})$.

(b) Show that $(A)$ is a constant-coefficient advection problem. Determine the splitting error we commit by computing, formally, $f^{n+1} = e^B e^A f^n$ instead of $f^{n+1} = e^{(A+B)} f^n$, $A$ and $B$ denoting the respective advection operators. Implement the most accurate solver from exercise 1 for both splitting steps.

(c) Solve system (2) for $T = 30$, $L_x = 12$, $v_{\text{max}} = -v_{\text{min}} = 5$ and the initial condition

\[ f(0, x, v) = f_0(x, v) = [1 + 0.01 \cos(2\pi x/L_x)] \frac{1}{(2\pi)^{1/2}} e^{-v^2/2}. \]

Choose an appropriate time step. Check for conservation of total mass, momentum, total energy and $L^2$-norm. Make a video of the evolution of $f$ in the $(x, v)$-plane. Plot the field energy $\int |E|^2/2 \, dx$ as a function of time. Does your result change when changing phase space resolution (more grid points)?

(d) Two-stream instability: set the velocity domain to $v \in [-10, 10]$ and run your VP-solver with the initial condition

\[ f(0, x, v) = \frac{1}{2} \left[ \frac{1}{\sqrt{2\pi}} e^{-(v-v_0)^2/2} + \frac{1}{\sqrt{2\pi}} e^{-(v+v_0)^2/2} \right] (1 + 10^{-3} \cos(k x)), \quad k = 0.2, \quad (3) \]

and with $L_x = 2\pi/k$, $T = 50$, $N_x = N_v = 256$. For the different stream velocities $v_0$ given below, plot the square root of the field energy $\int |E|^2 dx$ as a function of time and compare to the analytic growth $e^{\omega_i t}$, obtained from the dispersion relation:

\[ v_0 = 1.3 \quad (\omega_i = 0.0011), \quad v_0 = 2.4 \quad (\omega_i = 0.2258), \quad v_0 = 3.0 \quad (\omega_i = 0.2845). \]

(e) Bonus: Replace your finite difference Poisson solver by a spectral solver for $\partial_x E = 1 - \int f \, dv$ and repeat the numerical experiment.