1D elliptic problem with Galerkin FEM

Write a Galerkin finite element (FEM) solver for the elliptic equation
\[
\begin{align*}
- u''(x) + u(x) &= f(x), & x \in (a,b) \subset \mathbb{R}, \\
    u(a) = u(b) &= 0.
\end{align*}
\]
for given \( f \in C(\Omega) \) with \( \Omega = (a,b) \). We search for numerical solutions \( u_h \in V_h \), where \( V_h \subset V \) is a subspace of \( V = H^1_0(\Omega) \). For a Galerkin method, three main tasks need to be performed:

1. **Triangulation:** we compute a partition of the domain \([a,b]\) into \( M \) elements, where the vertices (interfaces between elements) are denoted by \( a = x_0, x_1, \ldots, x_M = b \). The \( i \)-th element is denoted by \( K_i = [x_i, x_{i+1}] \), \( i = 0, \ldots, M-1 \), its volume is \( C_i = x_{i+1} - x_i \).

2. **Basis functions:** recall that \( H^1(\Omega) \subset C^0(\Omega) \), such that the approximation space \( V_h \subset V \) consists of functions that are continuous on \((a,b)\). We choose \( V_h = X_h^r \), where \( X_h^r = \{ v_h \in C^0([a,b]) : v_h|_{K_i} \in P_r \ \forall \ i, \ v_h(a) = v_h(b) = 0 \} \),

3. **Solving the linear system arising from the weak formulation of (1) in the space \( V_h \).**

1. **Triangulation:** we compute a partition of the domain \([a,b]\) into \( M \) elements, where the vertices (interfaces between elements) are denoted by \( a = x_0, x_1, \ldots, x_M = b \). The \( i \)-th element is denoted by \( K_i = [x_i, x_{i+1}] \), \( i = 0, \ldots, M-1 \), its volume is \( C_i = x_{i+1} - x_i \).

A non-uniform partition can be created with the help of a positive spacing function \( H : [a,b] \rightarrow \mathbb{R}^+ \). The vertex \( x_i \) is the solution of
\[
c \int_a^{x_i} \frac{1}{H(x)} \, dx = i, \quad \text{with} \quad c = \frac{M}{\int_a^b 1/H(x) \, dx}.
\]
Setting \( g'(x) = c/H(x) \), \( g(a) = 0 \), the \( x_i \) are the roots of \( f_i := g(x) - i \). A second more convenient strategy is to compute preliminary vertices \( \tilde{x}_{i+1} = \tilde{x}_i + H(\tilde{x}_i) \) starting from \( \tilde{x}_0 = a \), until \( i + 1 = P \) with \( \tilde{x}_P > b \). Then \( M = P \) and we repeat the process as
\[
x_0 = a, \quad x_{i+1} = x_i + c H(x_i), \quad \text{with} \quad c = \frac{b-a}{\tilde{x}_P - a}.
\]

2. **Basis functions:** recall that \( H^1(\Omega) \subset C^0(\Omega) \), such that the approximation space \( V_h \subset V \) consists of functions that are continuous on \((a,b)\). We choose \( V_h = X_h^r \), where \( X_h^r = \{ v_h \in C^0([a,b]) : v_h|_{K_i} \in P_r \ \forall \ i, \ v_h(a) = v_h(b) = 0 \} \),

the space of piece-wise polynomials of degree \( r \), vanishing at the boundary. Construct bases \( (\varphi_i) \) for \( X_h^1 \) and \( X_h^2 \) with Lagrange polynomials and with splines, and for \( X_h^3 \) only with splines (a short introduction of how to use B-splines in Matlab can be found at [CompPlasmaPhys16 Exercise 5]). Map the basis functions to the reference element \([0,1]\) via
\[
R_i : [0,1] \rightarrow K_i, \quad \xi \mapsto x, \quad x = R_i(\xi) = x_i + \xi(x_{i+1} - x_i).
\]
3. **Linear solve**: write the weak formulation of problem (1) in the space $V_h$, by expanding the trial and the test function as

$$ u_h(x) = \sum_i u_i \varphi_i(x), \quad v_h = \sum_i v_i \varphi_i(x). $$

(6)

Compute the stiffness matrix $S = (s_{ij})$, the mass matrix $M = (m_{ij})$ and the right-hand side $b = (f_i)$,

$$ s_{ij} = \int_a^b \varphi'_i(x) \varphi'_j(x) \, dx, \quad m_{ij} = \int_a^b \varphi_i(x) \varphi_j(x) \, dx, \quad f_i = \int_a^b f(x) \varphi_i(x) \, dx. $$

(7)

where $i,j = 1, \ldots, \text{dim} \ X^r_h$. For $r \leq 2$, compute $(s_{ij})$ and $(m_{ij})$ analytically on the reference element. Save the matrices using the `sparse` command in Matlab. Then use a direct method to solve the linear system $(S + M)\hat{u} = b$ for $\hat{u} = (u_i)$.

Test the convergence rate w.r.t $h = \max(\mathcal{C}_i)$ on uniform ($\mathcal{H} = \text{const.}$) and non-uniform grids, in the $L^2$- and in the $H^1$-norm. As the reference use polynomials of the form $(x-a)(x-b)p(x)$, with polynomials $p(x)$ of degree 2, 4, 6 and 8. As a non-polynomial reference use the manufactured solution

$$ u_{ex}(x) = \sin(2\pi x/(b-a)) e^x. $$

(8)

Plot the results. As a quadrature rule for integrals, you could use e.g. Boole’s rule:

$$ \int_{y_1}^{y_5} f(x) \, dx = \frac{h}{90} \left[ 7f(y_1) + 32f(y_2) + 12f(y_3) + 32f(y_4) + 7f(y_5) \right] + O(h^7), $$

(9)

where $h = (y_5 - y_1)$ and $y_1, \ldots, y_5$ are five equally spaced points in the interval $[y_1, y_5]$. You can also try more or less accurate quadrature rules and see the influence on the error computation.

4. **Bonus**: construct new bases for $X^2_h$ on the reference element, $\xi \in [0,1]$, using the *shape functions*

(a) $\tilde{\varphi}_0 = (1 - \xi), \quad \tilde{\varphi}_1 = \xi, \quad \tilde{\varphi}_2 = (1 - \xi)\xi$ ,

(b) $\tilde{\varphi}_0 = 1, \quad \tilde{\varphi}_1 = \xi, \quad \tilde{\varphi}_2 = \xi^2$ .

Recall that continuity needs to be enforced at the vertices $(x_i)$!