TECHNISCHE UNIVERSITÄT MÜNCHEN
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Fallstudien der Mathematischen Modellbildung, part 3:
Asymptotic methods for perturbation problems, WS 2018/19
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Solutions

1. (a) The exact solutions are

$$x_\varepsilon = -\frac{\varepsilon}{2} \pm \sqrt{\frac{\varepsilon^2}{4} + 1}\,.$$

Taylor expansion leads to

$$x_I = 1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8} + \dots\,,$$

$$x_{II} = -1 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \dots\,.$$

(b) Substituting $x_\varepsilon = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ into the equation yields

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) - 1 = 0$$

Sorting in powers of ε and equating to zeros gives

$$\begin{aligned} x_0^2 - 1 &= 0\,, \\ 2x_0 x_1 + x_0 &= 0\,, \\ 2x_0 x_2 + x_1^2 + x_1 &= 0\,, \\ &\vdots \end{aligned}$$

from which we have

$$x_1 = -\frac{1}{2}\,, \qquad x_2 = \frac{x_0}{8}\,.$$

The two solutions $x_0 = \pm 1$ of the reduced problem lead thus to the correct expansions.

(c) Starting from $x^{(0)} = 1$ we obtain

$$x^{(1)} = \sqrt{1-\varepsilon}\,, \qquad x^{(2)} = \sqrt{1 - \varepsilon\sqrt{1-\varepsilon}}\,.$$

Using Taylor series expansion we obtain

$$\begin{aligned} x^{(2)} &= 1 - \varepsilon\frac{\sqrt{1-\varepsilon}}{2} - \varepsilon^2\frac{1-\varepsilon}{8} + \dots \\ &= 1 - \varepsilon\frac{1 - \frac{\varepsilon}{2} + \dots}{2} - \frac{\varepsilon^2}{8} + \dots \\ &= 1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8} + \dots\dots \end{aligned}$$

Starting from $x^{(0)} = -1$ we obtain

$$x^{(1)} = -\sqrt{1+\varepsilon}\,, \qquad x^{(2)} = -\sqrt{1+\varepsilon\sqrt{1+\varepsilon}}\,.$$

Using Taylor series expansion we obtain

$$\begin{aligned} x^{(2)} &= -1 - \varepsilon\frac{\sqrt{1+\varepsilon}}{2} + \varepsilon^2\frac{1+\varepsilon}{8} + \dots \\ &= -1 - \varepsilon\frac{1+\frac{\varepsilon}{2}+\dots}{2} + \frac{\varepsilon^2}{8} + \dots \\ &= -1 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \dots\dots \end{aligned}$$

2. (a) The exact solutions are

$$x_\varepsilon = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}\,.$$

Taylor expansion of the numerator gives

$$\sqrt{1+4\varepsilon} = 1 + 2\varepsilon - 2\varepsilon^2 + 4\varepsilon^3 + \dots\,,$$

which leads to the expansion of the roots:

$$\begin{aligned} x_I &= 1 - \varepsilon + 2\varepsilon^2 + \dots\,, \\ x_{II} &= -\frac{1}{\varepsilon} - 1 + \varepsilon + \dots\,. \end{aligned}$$

(b) Substituting $x_\varepsilon = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ into the equation yields

$$\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) - 1 = 0$$

Sorting in powers of ε and equating to zeros gives

$$\begin{aligned} x_0 - 1 &= 0\,, \\ x_0^2 + x_1 &= 0\,, \\ 2x_0x_1 + x_2 &= 0\,, \\ &\vdots \end{aligned}$$

from which we have

$$x_0 = 1\,, \qquad x_1 = -1\,, \qquad x_2 = 2\,.$$

The root x_I is thus obtained. Moreover, substituting $x_\varepsilon = x_{-1}/\varepsilon + x_0 + \varepsilon x_1 + \dots$ into the equation yields

$$\varepsilon\left(\frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \dots\right)^2 + \left(\frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \dots\right) - 1 = 0$$

Sorting in powers of ε and equating to zeros gives

$$\begin{aligned} x_{-1}^2 + x_{-1} &= 0\,, \\ 2x_{-1}x_0 + x_0 - 1 &= 0\,, \\ x_0^2 + 2x_{-1}x_1 + x_1 &= 0\,, \\ &\vdots \end{aligned}$$

The meaningful solution is

$$x_{-1} = -1\,, \qquad x_0 = -1 \qquad x_1 = 1\,,$$

from which we find the root x_{II}.

(c) To recover the first root we formulate the process

$$x^{(n+1)} = 1 - \varepsilon (x^{(n)})^2\,,$$

and start with $x^{(0)} = 1$ to obtain

$$x^{(1)} = 1 - \varepsilon\,, \qquad x^{(2)} = 1 - \varepsilon(1-\varepsilon)^2 = 1 - \varepsilon + 2\varepsilon^2 + \dots\,.$$

Moreover, to recover the second root we formulate the process

$$x^{(n+1)} = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x^{(n)}}\,,$$

and start with $x^{(0)} = -1/\varepsilon$ to obtain

$$x^{(1)} = -\frac{1}{\varepsilon} - 1\,, \qquad x^{(2)} = -\frac{1}{\varepsilon} - \frac{1}{1+\varepsilon} = -\frac{1}{\varepsilon} - 1 + \varepsilon \dots\,.$$

3. From de l'Hôpital's rule we have

$$\lim_{\varepsilon\to 0}(-\varepsilon \ln \varepsilon) = \lim_{\varepsilon\to 0}\left(-\frac{\ln\varepsilon}{\varepsilon^{-1}}\right) = \lim_{\varepsilon\to 0}\frac{1/\varepsilon}{1/\varepsilon^2} = \lim_{\varepsilon\to 0}\varepsilon = 0\,.$$

This shows that $-\varepsilon\ln\varepsilon \prec 1$ and $\varepsilon \prec -\frac{1}{\ln\varepsilon}$. Moreover, from

$$\lim_{\varepsilon\to 0}\left(-\frac{1}{\ln\varepsilon}\right) = 0\,,$$

it follows that $-\frac{1}{\ln\varepsilon} \prec 1$ and that $\varepsilon \prec -\varepsilon\ln\varepsilon$. Moreover, for $0 < \nu < 1$,

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\varepsilon^\nu} = \lim_{\varepsilon\to 0}\varepsilon^{1-\nu} = 0 \qquad \Rightarrow \qquad \varepsilon \prec \varepsilon^\nu\,,$$

and

$$\lim_{\varepsilon\to 0}\varepsilon(\ln\varepsilon)^2 = \lim_{\varepsilon\to 0}\frac{(\ln\varepsilon)^2}{\varepsilon^{-1}} = \lim_{\varepsilon\to 0}\left(-\frac{2\ln\varepsilon/\varepsilon}{1/\varepsilon^2}\right) = 2\lim_{\varepsilon\to 0}(-\varepsilon\ln\varepsilon) = 0$$

implies $-\varepsilon\ln\varepsilon \prec -\frac{1}{\ln\varepsilon}$. It remains to determine the place of ε^ν. Let's compute

$$\lim_{\varepsilon\to 0}\left(-\frac{\varepsilon\ln\varepsilon}{\varepsilon^\nu}\right) = \lim_{\varepsilon\to 0}\left(-\frac{\ln\varepsilon}{\varepsilon^{\nu-1}}\right) = \lim_{\varepsilon\to 0}\frac{1/\varepsilon}{(1-\nu)\varepsilon^{\nu-2}} = \frac{1}{1-\nu}\lim_{\varepsilon\to 0}\varepsilon^{1-\nu} = 0\,,$$

$$\lim_{\varepsilon\to 0}(-\varepsilon^\nu\ln\varepsilon) = \lim_{\varepsilon\to 0}\left(-\frac{\ln\varepsilon}{\varepsilon^{-\nu}}\right) = \lim_{\varepsilon\to 0}\frac{1/\varepsilon}{\nu\varepsilon^{-\nu-1}} = \frac{1}{\nu}\lim_{\varepsilon\to 0}\varepsilon^\nu = 0\,,$$

which tells us $-\varepsilon\ln\varepsilon \prec \varepsilon^\nu \prec -\frac{1}{\ln\varepsilon}$. In summary,

$$\varepsilon \prec -\varepsilon\ln\varepsilon \prec \varepsilon^\nu \prec -\frac{1}{\ln\varepsilon} \prec 1\,.$$

4. We work in the sup-norm:

(a) $$||\varphi_\varepsilon|| = \max_{A_1 \le x \le A_2} |\varphi(x,\varepsilon)| = |\varepsilon \ln(A_2/\varepsilon)|\,.$$

$$\lim_{\varepsilon\to 0} ||\varphi_\varepsilon|| = \lim_{\varepsilon\to 0} \left|\frac{\ln(A_2/\varepsilon)}{\varepsilon^{-1}}\right| = \lim_{\varepsilon\to 0} \left|\frac{1/\varepsilon}{\varepsilon^{-2}}\right| = \lim_{\varepsilon\to 0} \varepsilon = 0 \qquad \Rightarrow \qquad \varphi = o(1)\,.$$

(b) $$||\varphi_\varepsilon|| = \max_{A_1\varepsilon \le x \le A_2\varepsilon} |\varphi(x,\varepsilon)| = \varepsilon \underbrace{|\ln(A_2)|}_{=:k} \qquad \Rightarrow \qquad \varphi = O_{\mathrm{s}}(\varepsilon)\,.$$

(c) $$||\varphi_\varepsilon|| = \max_{A_1\varepsilon^2 \le x \le A_2\varepsilon^2} |\varphi(x,\varepsilon)| = |\varepsilon \ln(\varepsilon A_2)| \qquad \Rightarrow \qquad \varphi = o(1)\,.$$