TECHNISCHE UNIVERSITÄT MÜNCHEN

Zentrum Mathematik

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Fallstudien der Mathematischen Modellbildung, part 3:
Asymptotic methods for perturbation problems, WS 2018/19

http://www-m16.ma.tum.de/Allgemeines/Fallstd18

Übungsblatt 2 (January 16th, 2019)

Asymptotic expansions (AEs)

1. Let $x \in D = [0,1]$. Determine if the following approximations are uniformly valid as $\varepsilon \to 0$:

$$a) \quad \mathrm{e}^{\varepsilon x} = 1 + O(\varepsilon)\,,$$

$$b) \quad \frac{1}{x+\varepsilon} = O(1)\,,$$

$$c) \quad \mathrm{e}^{-x/\varepsilon} = o(\varepsilon^\nu)\,, \qquad \nu > 0\,.$$

2. For $\varepsilon \ll 1$, give a series expansion of the function

$$u(x,\varepsilon) = \frac{1}{1 + \varepsilon\frac{2x-1}{1-x}}\,.$$

Do we obtain an AE for $x \in D = [0,1]$? (Examine the order of the remainder in D when truncating the series after N terms.) Rewrite u as

$$u(x,\varepsilon) = \frac{1}{1 + \frac{\varepsilon}{1-x} - 2\varepsilon}$$

and deduce an AE that is uniformly valid in D.

3. From successive integration by parts, show that the function

$$E(t) = \int_t^\infty \frac{\mathrm{e}^{-s}}{s}\,\mathrm{d}s \quad \text{with } t > 0$$

has the following series expansion:

$$E(t) = \frac{\mathrm{e}^{-t}}{t}\left(1 - \frac{1}{t} + \frac{2}{t^2} + \ldots + (-1)^n\frac{n!}{t^n} + \ldots\right)\,.$$

Give an upper bound for the remainder $R_N(t)$ when truncating the series after N terms and show that one obtains in this way an AE of $E(t)$ for large values of t (as $t \to \infty$). Determine the limits

$$\lim_{N\to\infty,\, t\,\text{fixed}} R_N(t)\,, \qquad \lim_{t\to\infty,\, N\,\text{fixed}} R_N(t)\,,$$

and show that the series is divergent.

4. We are interested in the roots of the equation

$$\varepsilon x^2 + x - 1 = 0\,,$$

when ε is small (c.f. the second exercise from Übungsblatt 1). The reduced problem ($\varepsilon = 0$) gives only one root. The other root can be recovered by a change of variable of the form

$$y = \frac{x}{\delta(\varepsilon)}\,.$$

We assume that $0 < A_1 \le |y| \le A_2$ with constants A_1 and A_2 independent of ε. Consider the cases i) $\delta \prec 1$, ii) $\delta = 1$, iii) $1 \prec \delta \prec \varepsilon^{-1}$, iv) $\varepsilon^{-1} \prec \delta$ and v) $\delta = \varepsilon^{-1}$. Find the one significant choice of δ which allows us to determine the second root as $\varepsilon \to 0$.