We consider the dynamics of a charged particle in a static electromagnetic field \((E, B)\):

\[
\begin{align*}
\frac{dx(t)}{dt} &= v(t), \quad x(0) = x_0, \\
\frac{dv(t)}{dt} &= E(x(t)) + v(t) \times B(x(t)), \quad v(0) = v_0.
\end{align*}
\]

Here, \(x = xe_x + ye_y + ze_z\) in Cartesian position coordinates. The fields are given by \(B = Re_z\) and \(E = 2 \cdot 10^{-2} (xe_x + ye_y)/R^3\), where \(R = \sqrt{x^2 + y^2}\). They are related to the potentials \(A = R^2/3e_\xi\) and \(\phi = 2 \cdot 10^{-2}/R\) via the usual relations \(B = \nabla \times A\) and \(E = -\nabla \phi\), where \(e_\xi = (-y, x)^T/R\) is the normalized angular basis vector in cylindrical coordinates.

1. Implement a fourth-order Runge-Kutta scheme (RK4) to solve (1). For a system of ODEs \(\frac{dz}{dt} = F(z, t)\) with \(z : \mathbb{R} \to \mathbb{R}^d\) and vector field \(F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d\), the RK4 scheme \(z_n \mapsto z_{n+1}\) reads

\[
\begin{align*}
k_1 &= F(z_n, t_n), \\
k_2 &= F(z_n + \Delta t k_1/2, t_n + \Delta t/2), \\
k_3 &= F(z_n + \Delta t k_2/2, t_n + \Delta t/2), \\
k_4 &= F(z_n + \Delta t k_3, t_n + \Delta t),
\end{align*}
\]

\[z_{n+1} = z_n + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4).\]

Run the scheme with \(\Delta t = \pi/10\) (a twentieth of a cyclotron period) for \(n_{\text{max}} = 10^5\) time steps with initial conditions \(x_0 = (0, -1, 0)^T, \quad v_0 = (0.1, 0.01, 0)^T\).

Plot the solution in the \(xy\)-plane for \(n < 2300\) and for \(10^5 - 2300 < n < 10^5\). Check the conservation of the energy \(H = |v|^2/2 + \phi(x)\) in your simulation.

2. The (configuration space-) Lagrangian of the system (1) reads

\[L = \frac{|\dot{x}|^2}{2} + A(x) \cdot \ddot{x} - \phi(x).\]

Prove that the angular momentum \(p_\xi := R^2\dot{\xi} + R^3/3\) is a constant of the motion. Check this conservation in your simulations. Check also for the conservation of the magnetic moment \(\mu := (v_x^2 + v_y^2)/(2B(x))\).
3. **Volume preserving splitting methods.** We split the vector field \( \mathbf{F} = (\mathbf{v}, \mathbf{E} + \mathbf{v} \times \mathbf{B})^\top \) of the ODE system (1) into three parts, \( \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \), with
\[
\mathbf{F}_1 = (\mathbf{v}, 0)^\top, \quad \mathbf{F}_2 = (0, \mathbf{E}(\mathbf{x}))^\top, \quad \mathbf{F}_3 = (0, \mathbf{v} \times \mathbf{B}(\mathbf{x}))^\top.
\]
The flow associated to \( \mathbf{F}_i \) will be denoted by \( \varphi_i : \mathbb{R}^n \rightarrow \mathbb{R}^n \).

- For \( \mathbf{z} = (\mathbf{x}, \mathbf{v})^\top \), compute the exact flows \( \varphi_i \) of the ODEs
\[
d\mathbf{z}/dt = \mathbf{F}_i(\mathbf{z}), \quad \mathbf{z}(0) = \mathbf{z}_0.
\]
Hint: for \( i = 3 \), write \( \mathbf{v} \times \mathbf{b} = -\hat{\mathbf{b}} \mathbf{v} \) with the rotation matrix
\[
\hat{\mathbf{b}} = \begin{pmatrix}
0 & -b_3 & b_2 \\
b_3 & 0 & -b_1 \\
-b_2 & b_1 & 0
\end{pmatrix},
\]
and use \( \hat{\mathbf{b}}^3 = -\hat{\mathbf{b}} \) in the expansion of \( \exp(-B\hat{\mathbf{b}}) \).
- Implement a first-order splitting method \( \mathbf{z}_{n+1} = G_{1,\Delta t}^1 \mathbf{z}_n \) with
\[
G_{1,\Delta t}^1 = \varphi_{1,\Delta t}^1 \circ \varphi_{2,\Delta t}^2 \circ \varphi_{3,\Delta t}^3.
\]
- Implement a second-order splitting method \( \mathbf{z}_{n+1} = G_{2,\Delta t}^2 \mathbf{z}_n \) with
\[
G_{2,\Delta t}^2 = \varphi_{2,\Delta t}^{1/2} \circ \varphi_{3,\Delta t}^{3/2} \circ \varphi_{2,\Delta t}^{3/2} \circ \varphi_{2,\Delta t}^{3/2} \circ \varphi_{2,\Delta t}^{3/2}.
\]
- Implement a fourth-order splitting method \( \mathbf{z}_{n+1} = G_{4,\Delta t}^4 \mathbf{z}_n \) with
\[
G_{4,\Delta t}^4 = G_{2,\Delta t}^2 \circ G_{2,\Delta t}^2 \circ G_{2,\Delta t}^2, \quad \alpha = \frac{1}{2 - 2^{1/3}}, \quad \beta = 1 - 2\alpha.
\]
Perform the same simulation as for the RK4 scheme and compare the results, in particular the conservation properties of \( H, p_\xi \) and \( \mu \).