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# Numerical methods for hyperbolic systems

## Exercise sheet 2: Linear hyperbolic systems

**Exercice 1**

We considerthe wave equation

$$\left\{\begin{array}{l} \partial_t p + \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \end{array}\right. \tag{1}$$

**1.** Diagonalize the system and show that the upwind scheme is given by

$$\left\{\begin{array}{l} \dfrac{v_j^{n+1} - v_j^n}{\Delta t} + \dfrac{v_j^n - v_{j-1}^n}{\Delta x} = 0, \\ \\ \dfrac{w_j^{n+1} - w_j^n}{\Delta t} - \dfrac{w_{j+1}^n - w_j^n}{\Delta x} = 0, \end{array}\right. \tag{2}$$

with $v = p + u$ and $w = p - u$.

**2.** Prove that the upwind scheme (2) for the initial system (1) can be write on the following form

$$\left\{\begin{array}{l} \dfrac{p_j^{n+1} - p_j^n}{\Delta t} + \dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \dfrac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{2\Delta x} = 0, \\ \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + \dfrac{p_{j+1}^n - p_{j-1}^n}{2\Delta x} - \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\Delta x} = 0, \end{array}\right. \tag{3}$$

and used this form to compute the consistency error.

**3.** Prove that the scheme (2) satisfy the maximum principle under a CFL condition.

**4.** Prove that the scheme (2) is stable for all $l^q$ norms ($1 \le q \le \infty$) using the previous result and convex functions. The $l^q$ norm is defined by

$$||(v,w)||_{l^q} = \left(\Delta x \sum_j ||(v_j,w_j)||_q^q\right)^{\frac{1}{q}}$$

with $||(v_j,w_j)||_q^q = |v_j|^q + |w_j|^q$.

**Additional question**

We introduce the damped wave equation

$$\begin{cases} \partial_t p + \partial_x u = 0, \\ \partial_t u + \partial_x p = -\sigma u, \end{cases} \tag{4}$$

and the upwind scheme associated

$$\begin{cases} \dfrac{p_j^{n+1} - p_j^n}{\Delta t} + \dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \dfrac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{2\Delta x} = 0, \\ \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + \dfrac{p_{j+1}^n - p_{j-1}^n}{2\Delta x} - \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\Delta x} = -\sigma u_j^n, \end{cases} \tag{5}$$

**5.** We call "steady states" the solutions of the systems defined by $\partial_x u = 0$ and $\partial_x p = -\sigma u$. Prove that (5) preserve exactly the steady states.

**Exercice 2**

We consider the Maxwell equation

$$\begin{cases} \mu \partial_t B + c \partial_x E = -c\sigma^* B, \\ \varepsilon \partial_t E + c \partial_x B = -c\sigma E, \end{cases} \tag{6}$$

with periodic boundary condition on $\Omega = [0, L]$ and $\mu > 0$, $\varepsilon > 0$, $\sigma > 0$, $\sigma^* > 0$, $c > 0$.

**1.** Prove the following energy equality and the uniqueness of the solutions.

$$\frac{d}{dt}\left( \int_\Omega \varepsilon |E(t,x)|^2 + \mu |B(t,x)|^2 dx \right) = -\int_\Omega \sigma |E(t,x)|^2 + \sigma^* \mu |B(t,x)|^2 dx \tag{7}$$

**2.** We introduce the plane waves (which are a good approximations of physical waves) defined by $E(t,x) = E_0 e^{j(wt-kx)}$ and $B(t,x) = B_0 e^{j(wt-kx)}$ ($j$ complex number) with $E_0 \in \mathbb{R}$, $B_0 \in \mathbb{R}$, $k$ the wave vector and $w$ the frequency. Give the conditions (called dispersion relation) on $w$ and $k$ such as the plane waves are solutions of (6) for $\sigma = 0$ and $\sigma^* = 0$.

In this part we assume that $\sigma = 0$ and $\sigma^* = 0$. Now we introduce the DG centered scheme for (6). The mesh $\Omega_h$ is defined by $n+1$ points $x_i$ and $n$ cells $K_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$. The volume of the cell $K_i$ is $\Delta x_i = |x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}|$. We call a generic cell $K$. To finish the test function are defined by $v \in V_h = \{v / v|_K \in \mathbb{P}^p(K)\}$. The scheme is given by

$$\begin{cases} \varepsilon \displaystyle\sum_{l=0}^{k} \int_{K_i} \phi_l^i \phi_m^i \left( \frac{E_{l,i}^{n+1} - E_{l,i}^n}{\Delta t} \right) - c \sum_{l=0}^{k} B_{l,i}^n \int_{K_i} \phi_l^i \partial_x \phi_m^i + c \sum_{l=0}^{k} \left[ B\phi_m^i \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \quad \forall 0 \le m \le k, \\ \mu \displaystyle\sum_{l=0}^{k} \int_{K_i} \phi_l^i \phi_m^i \left( \frac{B_{l,i}^{n+1} - B_{l,i}^n}{\Delta t} \right) - c \sum_{l=0}^{k} E_{l,i}^n \int_{K_i} \phi_l^i \partial_x \phi_m^i + c \sum_{l=0}^{k} \left[ E\phi_m^i \right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = 0, \quad \forall 0 \le m \le k, \end{cases} \tag{8}$$

with $\left[B\phi_m^i\right]_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} = \frac{1}{2}\left( B_{l,i+1}^n \phi_l^{i+1}(x_{i+\frac{1}{2}}) \phi_m^i(x_{i+\frac{1}{2}}) + B_{l,i}^n \phi_l^i(x_{i+\frac{1}{2}}) \phi_m^i(x_{i+\frac{1}{2}}) \right)$

$-\frac{1}{2}\left(B^n_{l,i-1}\phi^{i-1}_l(x_{i-\frac{1}{2}})\phi^i_m(x_{i-\frac{1}{2}}) + B^n_{l,i}\phi^i_l(x_{i-\frac{1}{2}})\phi^i_m(x_{i-\frac{1}{2}})\right).$

**3.** We consider $V_h = P^1(K)$. We propose to use the Lagrange polynomial associated with the point $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$. Prove that the family is a basis of $V_h$. Write the scheme in a cell $K_i$.

**4.** In this exercise we propose to study the numerical dispersion relation which define the numerical wave vector $\tilde{k}$ for $V_h = P^0(K) = \text{Span}(1)$. Write the scheme for this basis. Now we define $B^n_i = B_0 e^{j(wn\Delta t - \tilde{k}i\Delta x)}$ and $E^n_i = E_0 e^{j(wn\Delta t - \tilde{k}i\Delta x)}$ with $j$ the complex number and $i$ the index of the cell. Gives the relation between $w$ and $\tilde{k}$ such as the discrete plane waves are solutions of (8). Show that the numerical dispersive relation is $\tilde{k}^2 = \frac{w^2}{c^2} + O(\Delta x^p + \Delta t^q)$ with $p > 1$ and $q > 1$.