TU München
Zentrum Mathematik
Lehrstuhl M16
E. Sonnendrücker
A. Ratnani

Numerical methods for hyperbolic systems

Exercise sheet 3: Mixed Finite Element discretization

Exercice 1 Let $\Omega \subset \mathbb{R}^2$ be a smooth domain with a partition Ω_1 and Ω_2, both with non-zero measure. Let $\mathbf{H}$ be a vector field in $\mathbb{R}^2$ and E a scalar field. We define a $2D$ rotational operator for vector and scalar fields as $\nabla \times \mathbf{H} = \partial_x \mathbf{H}_y - \partial_y \mathbf{H}_x$ and $\nabla \times E = \begin{pmatrix} \partial_y E \\ -\partial_x E \end{pmatrix}$.
Let $\mu, \sigma > 0$ and $j \in L^2(\Omega)$. We consider the following problem

$$\begin{cases} \mu \mathbf{H} & = -\nabla \times E & , \Omega \\ \nabla \times \mathbf{H} & = 0 & , \Omega_1 \\ \nabla \times \mathbf{H} & = \sigma E + j & , \Omega_2 \\ \mathbf{H} \times \mathbf{n} & = 0 & , \partial\Omega \end{cases} \tag{1}$$

1. Give a weak formulation for this problem in the form of a saddle-point problem in $H_0(curl, \Omega) \times L^2(\Omega_1)$.

2. For $e \in L^2(\Omega_1)$, prove that there exists $\tilde{e} \in L^2_{\int=0}(\Omega)$ such that $\tilde{e}|_{\Omega_1} = e$ and $\|\tilde{e}\|_{0,\Omega} \leq \|e\|_{0,\Omega_1}$ for a given constant $c > 0$.

3. Prove that $\exists c > 0$ such that

$$\forall e \in L^2(\Omega_1), \sup\{\frac{(\nabla \times \mathbf{b}, e)}{\|\mathbf{b}\|_{H(curl,\Omega)}}, \mathbf{b} \in H_0(curl, \Omega)\} \geq c\|e\|_{0,\Omega_1}$$

(*hint:* solve $-\nabla^2 \phi = \tilde{e}, \partial_n \phi|_{\partial\Omega} = 0$ and set $\mathbf{b} = \nabla \times \phi$)

4. Prove that the weak problem derived in question **1.** is well-posed.

5. Propose a pair of finite elements to solve this problem.

Exercice 2 Let $\Omega \subset \mathbb{R}^2$ be a smooth domain with a partition Ω_1 and Ω_2, both with non-zero measure. Let $\sigma \geq 0$, $\mathbf{f} \in [H^{-1}(\Omega)]^d$ and $g \in L^2(\Omega_2)$.

1. Write a weak formulation in $[H^{-1}(\Omega)]^d \times L^2(\Omega_2)$ for the problem

$$\begin{cases} -\nabla^2 \mathbf{u} + \nabla p & = \mathbf{f} & , \Omega \\ \nabla \cdot \mathbf{u} & = 0 & , \Omega_1 \\ \sigma \nabla \cdot \mathbf{u} + p & = g & , \Omega_2 \\ \mathbf{u} & = 0 & , \partial\Omega \end{cases} \tag{2}$$

2. For $q \in L^2(\Omega_1)$, prove that there exists $\tilde{q} \in L^2_{\int=0}(\Omega)$ such that $\tilde{q}|_{\Omega_1} = q$ and $\|\tilde{q}\|_{0,\Omega} \leq \|q\|_{0,\Omega_1}$ for a given constant $c > 0$.

3. Prove that $\exists c > 0$ such that

$$\forall q \in L^2(\Omega_1), \sup\{\frac{(\nabla \cdot \mathbf{v}, q)_{0,\Omega_1}}{\|\mathbf{v}\|_{1,\Omega}}, \ \mathbf{v} \in [H_0^1(\Omega)]^d\} \geq c\|q\|_{0,\Omega_1}$$

4. Prove that the weak problem derived in question **1.** is well-posed

5. Propose a pair of finite elements so solve this problem.