TU München
Zentrum Mathematik
Lehrstuhl M16
E. Sonnendrücker
A. Ratnani

Numerical methods for hyperbolic systems

Exercise sheet 4: Nonlinear scalar equations

Exercice 1 We consider the Burgers equation

$$\begin{cases} \partial_t u + \partial_x \left(\dfrac{u^2}{2}\right) = 0, \quad \forall x \in \mathbb{R}, \quad t > 0, \\ u(t=0,x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \end{cases} \tag{1}$$

with $u_L \leq u_R$

1. Prove that

$$u(t,x) = \begin{cases} u_L, & \frac{x}{t} < 0.5(u_L + u_R), \\ u_R, & \frac{x}{t} > 0.5(u_L + u_R), \end{cases} \tag{2}$$

and

$$u(t,x) = \begin{cases} u_L, & \frac{x}{t} < u_L, \\ \frac{x}{t}, & u_L < \frac{x}{t} < u_R, \\ u_R, & \frac{x}{t} > u_R, \end{cases} \tag{3}$$

are weak solutions of (1).

2. We define the entropy $\eta(u) = \frac{u^{2p}}{2p} + \alpha \frac{u^2}{2}$ ($\alpha > 0$, $p > 2$) associated with (1) and the entropic flux associated with $\xi(u) = \frac{u^{2p+1}}{2p+1} + \alpha \frac{u^3}{3}$. Prove that the function (2) is not a weak entropy solution and the function (3) is a weak entropy solution. Give a condition on u_L and u_R such as (2) is a weak entropy solution.
Corollary useful: for the equation $\partial_t u + \partial_x f(u) = 0$, if f is convexe a shock is entropic if $f^{'}(u_L) > \sigma > f^{'}(u_R)$.

Exercice 2

Firstly we consider the equation (1) on the non-conservative form

$$\partial_t u + u \partial_x u = 0, \quad \forall x \in \mathbb{R}, \quad t > 0, \tag{4}$$

We propose to approximate (4) with the finite volumes scheme

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{a_j^n}{\Delta x}(u_j^n-u_{j-1}^n)=0, \tag{5}$$

where the discrete velocity is given by $a_j^n=u_j^n$, $a_j^n=u_{j-1}^n$ or $a_j^n=\frac{u_j^n+u_{j-1}^n}{2}$.
1. Discussing the conservativity of the scheme for the different discrete velocities.
Now we consider a nonlinear scalar equation

$$\begin{cases} \partial_t u+\partial_x f(u)=0, \quad \forall x\in\mathbb{R}, \quad t>0, \\ u(t=0,x)=u^0(x), \end{cases} \tag{6}$$

and the following scheme

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{f_{j+\frac{1}{2}}^n-f_{j-\frac{1}{2}}^n}{\Delta x}=0, \tag{7}$$

with $f_{j+\frac{1}{2}}^n=\frac{1}{2}(f(u_{j+1}^n)+f(u_j^n))+\frac{c}{2}(u_j^n-u_{j+1}^n)$, $m=\min_x u_0(x)$, $M=\max_x u_0(x)$ and $\max_{m\le x\le M}|f'(x)|\le c$.

2. Prove that the scheme satisfy the maximum principle under a CFL condition.

Additional questions

Now we propose to prove that the scheme is entropic which correspond to satisfy

$$\frac{\eta(u_j^{n+1})-\eta(u_j^n)}{\Delta t}+\frac{\xi_{j+\frac{1}{2}}^n-\xi_{j-\frac{1}{2}}^n}{\Delta x}\le 0,$$

with $(\eta(u),\xi(u))$ a couple entropy-entropic flux and $\xi_{j+\frac{1}{2}}^n$ the numerical entropic flux

$$\xi_{j+\frac{1}{2}}^n=\frac{\xi(u_{j+1}^n)+\xi(u_j^n)}{2}+\frac{c}{2}(\eta(u_j^n)-\eta(u_{j+1}^n)).$$

3. Prove that

$$\frac{\eta(u_j^{n+1})-\eta(u_j^n)}{\Delta t}+\frac{\xi_{j+\frac{1}{2}}^n-\xi_{j-\frac{1}{2}}^n}{\Delta x}\le\frac{1}{2}(\phi(u_{j+1}^n)+\psi(u_{j-1}^n))$$

with

$\phi(z)=\nu\left(u_j^n+\frac{\Delta t}{\Delta x}c(z-u_j^n)-\frac{\Delta t}{\Delta x}(f(z)-f(u_j^n))\right)-\eta(u_j^n)-\frac{\Delta t}{\Delta x}c(\eta(z)-\eta(u_j^n))+\frac{\Delta t}{\Delta x}(\xi(z)-\xi(u_j^n))$,

and

$\psi(z)=\nu\left(u_j^n+\frac{\Delta t}{2\Delta x}c(-u_j^n+z)-\frac{\Delta t}{2\Delta x}(f(u_j^n)-f(z))\right)-\eta(u_j^n)-\frac{\Delta t}{2\Delta x}c(-\eta(u_j^n)+\eta(z))+\frac{\Delta t}{2\Delta x}(\xi(u_j^n)-\xi(z))$.

4. Prove that $\psi(z)\le 0$, $\phi(z)\le 0$ under the CFL $\frac{c\Delta t}{2\Delta x}<1$ and conclude.

Additional exercise We consider a linear hyperbolic system with stiff nonlinear source term.

$$
\begin{cases} \partial_t p + \partial_x u = 0, \\ \partial_t u + a\partial_x p = \dfrac{1}{\varepsilon}(f(p) - u), \end{cases} \tag{8}
$$

with $\sqrt{a} \geq |f^{'}(p)|$.

1. Formally prove that when ε tends to zero, the system (8) tends to $\partial_t p + \partial_x f(p) = 0$.

Idea : Try to obtain $\partial_t u + \partial_x f(u) = \varepsilon \partial_x \left[(a - f^{'}(p)^2)\partial_x p \right] + o(\varepsilon^2)$.

2. We propose the splitting scheme (9)-(10)

$$
\begin{cases} \dfrac{p_j^{n+\frac{1}{2}} - p_j^n}{\Delta t} = 0, \\ \dfrac{u_j^{n+\frac{1}{2}} - u_j^n}{\Delta t} = \dfrac{1}{\varepsilon}(f(p_j^n) - u_j^n). \end{cases} \tag{9}
$$

$$
\begin{cases} \dfrac{p_j^{n+1} - p_j^{n+\frac{1}{2}}}{\Delta t} + \dfrac{u_{j+1}^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}}}{\Delta x} - \dfrac{\sqrt{a}\Delta x}{2} \dfrac{p_{j+1}^{n+\frac{1}{2}} - 2p_j^{n+\frac{1}{2}} + p_{j-1}^{n+\frac{1}{2}}}{\Delta x^2} = 0, \\ \dfrac{u_j^{n+1} - u_j^{n+\frac{1}{2}}}{\Delta t} + \dfrac{p_{j+1}^{n+\frac{1}{2}} - p_{j-1}^{n+\frac{1}{2}}}{\Delta x} - \dfrac{\sqrt{a}\Delta x}{2} \dfrac{u_{j+1}^{n+\frac{1}{2}} - 2u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}}{\Delta x^2} = 0. \end{cases} \tag{10}
$$

Assuming that $u_j^0 = f(p_n^0) + \varepsilon$ (the initial data are close to the equilibrium). Explain why this scheme is not adapted to treat the system (8) with big time step.

3. Propose a modification of the splitting scheme to obtain a better accuracy for big time step and justify your modification.