Mixed Finite Elements Method

A. Ratnani\textsuperscript{34}, E. Sonnendrücker\textsuperscript{34}

\textsuperscript{3}Max-Planck Institut für Plasmaphysik, Garching, Germany
\textsuperscript{4}Technische Universität München, Garching, Germany

Contents

1 Introduction 2
  1.1 Notations ............................................. 2

2 Poisson equation 2

3 Stokes equation 4
  3.1 First weak formulation ................................ 4
  3.2 Second weak formulation ............................... 4

4 Time Harmonic Maxwell problem 5

5 Time Domain Maxwell problem 6
  5.1 First formulation ..................................... 6
  5.2 Second formulation ................................. 6

6 Exercises 7
1 Introduction

1.1 Notations

Scalar-valued test functions will be denoted by \( \phi \), while \( \phi_i \) will denote a scalar basis function, after reordering all the basis functions of a given discrete space.

Vector-valued functions will be written in bold, like \( \mathbf{u}, \mathbf{v}, \mathbf{Ψ} \). \( \Psi \) will denote a vector-valued test function, while \( \Psi_i \) will denote a vector-valued basis function (after reordering the basis functions).

Even if most of what follows holds for both the 2D and 3D cases, we will restrict our studies to the 2D one. We recall that in 2D, there are two curl operators, one acting on scalars \( \nabla \times \phi = (\partial_y \phi, -\partial_x \phi) \) and one acting on vectors \( \nabla \times \mathbf{Ψ} = \partial_x \mathbf{Ψ} - \partial_y \mathbf{Ψ} \). Differential operators that return a vector (grad, curl) will be written in bold \((\nabla, \nabla \times)\).

We also recall the Green formula for the divergence and curl/rotational operators

\[
\int_{\Omega} (\nabla \cdot F) G = -\int_{\Omega} F \cdot \nabla G + \int_{\partial\Omega} (F \cdot n) G, \quad \forall F \in H(\text{div}, \Omega), \forall G \in H^1(\Omega)
\]

\[
\int_{\Omega} (\nabla \times F) \cdot G = \int_{\Omega} G \nabla \times F - \int_{\partial\Omega} (G \times n) \cdot F, \quad \forall F \in H(\text{curl}, \Omega), \forall G \in H^1(\Omega)
\]

If \( \Omega \subset \mathbb{R}^d, \mathbf{u} = (u_1, u_2, \ldots, u_d) \) and \( \mathbf{Ψ} = (\Psi_1, \Psi_2, \ldots, \Psi_d) \), we recall the notation

\[
\int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{Ψ} = \sum_{i=1}^{d} \int_{\Omega} \nabla u_i \cdot \nabla \Psi_i
\]

2 Poisson equation

Let’s start with the Poisson problem

\[
\begin{align*}
-\nabla^2 \phi &= f & \text{in } \Omega, \\
\phi &= 0 & \text{on } \partial\Omega,
\end{align*}
\]

As we know, the associated variational formulation can be written as a minimization problem over \( H^1(\Omega) \). Let us now, introduce an auxiliary variable \( \mathbf{u} = \nabla \phi \). The Poisson problem is then equivalent to the following system

\[
\begin{align*}
-\nabla \cdot \mathbf{u} &= f & \text{in } \Omega, \\
\mathbf{u} &= \nabla \phi & \text{in } \Omega, \\
\phi &= 0 & \text{on } \partial\Omega,
\end{align*}
\]

In order to have a weak formulation, we need to multiply the first equation by a scalar-valued test function \( \phi \), while the second one will be multiplied by a vector-valued test function \( \mathbf{Ψ} \). Therefor,

\[
\begin{align*}
-\int_{\Omega} (\nabla \cdot \mathbf{u}) \phi &= \int_{\Omega} f \phi, \\
\int_{\Omega} \mathbf{u} \cdot \mathbf{Ψ} &= \int_{\Omega} \nabla \phi \cdot \mathbf{Ψ},
\end{align*}
\]
Now let us use the Green’s formula for the divergence. Note that the integration by parts will only be done on the second equation. This leads to the following weak formulation

\[
\begin{align*}
\int_{\Omega} (\nabla \cdot \mathbf{u}) \varphi + \int_{\Omega} f \varphi &= 0, \\
\int_{\Omega} \mathbf{u} \cdot \mathbf{\Psi} + \int_{\Omega} \phi \nabla \cdot \mathbf{\Psi} &= 0,
\end{align*}
\]  

(7)

Now, let us consider the bilinear form

\[
B((\mathbf{u}, \phi), (\mathbf{\Psi}, \varphi)) = \int_{\Omega} (\nabla \cdot \mathbf{u}) \varphi + \left( \int_{\Omega} \mathbf{u} \cdot \mathbf{\Psi} + \int_{\Omega} \phi \nabla \cdot \mathbf{\Psi} \right)
\]

(8)

Where we consider the trial vector-valued function \((\mathbf{u} \phi)\) and the test vector-valued function \((\mathbf{\Psi} \varphi)\).

For the sake of clarity, we rewrite the bilinear form as the following

\[
B((\mathbf{u}, \phi), (\mathbf{\Psi}, \varphi)) = \int_{\Omega} \mathbf{u} \cdot \mathbf{\Psi} + \left( \int_{\Omega} (\nabla \cdot \mathbf{u}) \varphi + \int_{\Omega} \phi \nabla \cdot \mathbf{\Psi} \right)
\]

(9)

Therefor, it is easy to see that our bilinear form is symmetric.

Up to now, we didn’t mention what are the correct spaces for this formulation. Because, there is no differential operator over the unknown \(\phi\) and the scalar-valued test function \(\varphi\), these functions can be placed in \(L^2(\Omega)\). On the other hand, both \(\mathbf{u}, \mathbf{\Psi}, \nabla \cdot \mathbf{u}\) and \(\nabla \cdot \mathbf{\Psi}\) must be in \(L^2(\Omega)\).

Therefor, we need the Hilbert space \(H(\text{div}, \Omega)\) defined as

\[
H(\text{div}, \Omega) = \{ \mathbf{\Psi} \in L^2(\Omega), \ \nabla \cdot \mathbf{\Psi} \in L^2(\Omega) \}
\]

(10)

Therefor, the mixed weak formulation for the Poisson equation is

Find \((\mathbf{u}, \phi) \in H(\text{div}, \Omega) \times L^2(\Omega)\) such that \(\nabla (\mathbf{\Psi}, \varphi) \in H(\text{div}, \Omega) \times L^2(\Omega)\)

\[
\begin{align*}
\int_{\Omega} \mathbf{u} \cdot \mathbf{\Psi} + \int_{\Omega} \phi \nabla \cdot \mathbf{\Psi} &= 0, \\
\int_{\Omega} (\nabla \cdot \mathbf{u}) \varphi &= -\int_{\Omega} f \varphi,
\end{align*}
\]

(11)

Now let us introduce the bilinear form \(a(\mathbf{u}, \mathbf{\Psi}) = \int_{\Omega} \mathbf{u} \cdot \mathbf{\Psi}, \ \forall \mathbf{u}, \mathbf{\Psi} \in H(\text{div}, \Omega)\) and \(b(\phi, \mathbf{\Psi}) = \int_{\Omega} \phi \nabla \cdot \mathbf{\Psi}, \ \forall \phi, \mathbf{\Psi} \in L^2(\Omega) \times H(\text{div}, \Omega)\). The later problem can be written as

Find \((\mathbf{u}, \phi) \in H(\text{div}, \Omega) \times L^2(\Omega)\) such that \(\nabla (\mathbf{\Psi}, \varphi) \in H(\text{div}, \Omega) \times L^2(\Omega)\)

\[
\begin{align*}
a(\mathbf{u}, \mathbf{\Psi}) + b(\phi, \mathbf{\Psi}) &= r_1(\mathbf{\Psi}), \\
b(\phi, \mathbf{u}) &= r_2(\varphi),
\end{align*}
\]

(12)
3 Stokes equation

Let \( u \) be the velocity of a fluid and \( p \) the pressure in the fluid. We denote by \( f \) the body forces. The Stokes problem writes

\[
\begin{align*}
-\nabla^2 u + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= g \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

When \( g = 0 \), the fluid is said to be incompressible. Because \( \int_\Omega \nabla \cdot u = \int_{\partial \Omega} u \cdot n = 0 \), the source term \( g \) must fulfill the compatibility condition \( \int_\Omega g = 0 \). Therefore, the function \( g \) will be in the space \( L^2_0(\Omega) \), defined as

\[
L^2_0(\Omega) = \{ \varphi \in L^2(\Omega); \int_\Omega \varphi = 0 \}
\]

Solving the Stokes equation consists of looking for \( u \in H^1(\Omega) \) and \( p \in L^2_0(\Omega) \), where \( f \in H^{-1}(\Omega) \) and \( g \in L^2_0(\Omega) \).

3.1 First weak formulation

The first formulation consists on putting the constraint \( \nabla \cdot u = 0 \) directly into the continuous (and discrete) space. We start by introducing the space

\[
H_0(\text{div}, \Omega) = \{ \Psi \in H^1_0(\Omega), \nabla \cdot \Psi = 0 \}
\]

Find \( u \in V \) such that \( \forall \Psi \in H_0(\text{div}, \Omega) \)

\[
\int_\Omega \nabla u : \nabla \Psi = \int_\Omega f \cdot \Psi
\]

This bilinear form can be shown to be coercive and one can directly apply the Lax-Milgram theorem. However, this formulation has a major drawback from the practical point of view: the pressure is not solved directly. Moreover, on the discrete level, we need to construct an increasing sequence \( (V_h)_h \) of subspaces of \( V \). This can be done thanks to the discrete DeRham sequence.

3.2 Second weak formulation

Now let us keep the space for \( u \) as big as possible (which means the space \( H^1_0(\Omega) \)) and then add the constraint \( \nabla \cdot u = 0 \), as a Lagrange multiplier. In this case, the problem writes

Find \( (u, p) \in H^1_0(\Omega) \times L^2_0(\Omega) \) such that \( \forall (\Psi, \varphi) \in H^1_0(\Omega) \times L^2_0(\Omega) \)

\[
\begin{align*}
\int_\Omega \nabla u : \nabla \Psi - \int_\Omega p \nabla \cdot \Psi &= \int_\Omega f \cdot \Psi \\
\int_\Omega \varphi \nabla \cdot u &= 0
\end{align*}
\]
Now let us introduce the bilinear form \( a(u, \Psi) = \int_{\Omega} \nabla u : \nabla \Psi \), \( \forall u, \Psi \in H(\text{div}, \Omega) \) and \( b(p, \Psi) = -\int_{\Omega} p \nabla \cdot \Psi, \forall (p, \Psi) \in L^2(\Omega) \times H(\text{div}, \Omega) \). The later problem can be written as

\[
\text{Find } (u, p) \in H^1_0(\Omega) \times L^2_0(\Omega) \text{ such that } \forall (\Psi, \varphi) \in H^1_0(\Omega) \times L^2_0(\Omega)
\begin{align*}
a(u, \Psi) + b(p, \Psi) &= r_1(\Psi), \\
b(\varphi, u) &= r_2(\varphi),
\end{align*}
\]

(18)

4 Time Harmonic Maxwell problem

The Time Harmonic Maxwell problem writes

\[
\text{Find } E \neq 0 \text{ such that } \begin{cases}
\nabla \times \nabla \times E = \omega^2 E & \text{in } \Omega, \\
E \times n = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(19)

Two cases occur: either \( \omega \) is provided and then we only look for \( E \). The other case is known as an eigenvalue problem, and the problem must be solved for both \( E \) and \( \omega \).

If we multiply by a vector-valued function \( \Psi \), and integrating by parts using the curl/rotational Green formula, we get

\[
\int_{\Omega} \nabla \times E \cdot \nabla \times \Psi = \omega^2 \int_{\Omega} E \cdot \Psi
\]

(20)

Again, the space where \( E \) lives was not mentioned; let us find what this space can be.

We need both \( E, \Psi, \nabla \times E \) and \( \nabla \times \Psi \) to be in \( L^2(\Omega) \), while satisfying the boundary condition \( E \times n = 0 \) on the boundary \( \partial \Omega \).

Therefore, we need the Hilbert space \( H_0(\text{curl}, \Omega) \) defined as

\[
H_0(\text{curl}, \Omega) = \{ \Psi \in H(\text{curl}, \Omega), \Psi \times n = 0 \text{ on } \partial \Omega \}
\]

(21)

where

\[
H(\text{curl}, \Omega) = \{ \Psi \in L^2(\Omega), \nabla \times \Psi \in L^2(\Omega) \}
\]

(22)

Therefore, in weak form, the problem writes

\[
\text{Find } E \in H_0(\text{curl}, \Omega), E \neq 0 \text{ such that } \forall \Psi \in H_0(\text{curl}, \Omega)
\int_{\Omega} \nabla \times E \cdot \nabla \times \Psi = \omega^2 \int_{\Omega} E \cdot \Psi
\]

(23)

5
5 Time Domain Maxwell problem

The Maxwell equations write

\[
\begin{aligned}
\nabla \cdot D &= \rho \\
\nabla \cdot B &= 0 \\
\partial_t B &= -\nabla \times E \\
\partial_t D &= \nabla \times H - J
\end{aligned}
\]

(24)

In the case of linear, isotropic and non-dispersive materials, we have two additional relations: \( B = \mu H \) and \( D = \epsilon E \). Furthermore, we assume that \( J = \sigma E \). Plugging these relations in (Eq. 24), we get

\[
\begin{aligned}
\nabla \cdot (\epsilon E) &= \rho \\
\nabla \cdot (\mu H) &= 0 \\
\mu \partial_t H &= -\nabla \times E \\
\epsilon \partial_t E &= \nabla \times H - J
\end{aligned}
\]

(25)

Now, let us assume that both \( \epsilon \) and \( \mu \) are constants and equal to 1. We also, restrict our study to the 2D case and we consider the Transverse Electric (TE) mode. Therefore, the problem is only involving the variables \( E_x, E_y \) and \( H_z \). In this case, the Time Domain Maxwell problem writes

\[
\begin{aligned}
\partial_t H &= -\nabla \times E \\
\partial_t E &= \nabla \times H - J \\
\nabla \cdot E &= \rho
\end{aligned}
\]

(26)

where we consider here that \( E = (E_x, E_y) \) and \( H = H_z \).

5.1 First formulation

The first variational formulation, in the case of perfectly conducting boundary conditions, can be derived using the curl/rotational Green formula in Ampere’s law. Therefore, the TE Maxwell problem reads

\[
\begin{aligned}
\frac{d}{dt} \int_{\Omega} E \cdot \Psi - \int_{\Omega} H \nabla \times \Psi &= - \int_{\Omega} J \cdot \Psi \\
\frac{d}{dt} \int_{\Omega} H \varphi + \int_{\Omega} (\nabla \times E) \varphi &= 0
\end{aligned}
\]

(27)

5.2 Second formulation

Using the divergence Green formula in Faraday’s law yields the second variational formulation.

\[
\begin{aligned}
\frac{d}{dt} \int_{\Omega} E \cdot \Psi - \int_{\Omega} (\nabla \times H) \cdot \Psi &= - \int_{\Omega} J \cdot \Psi \\
\frac{d}{dt} \int_{\Omega} H \varphi + \int_{\Omega} E \cdot (\nabla \times \varphi) &= 0
\end{aligned}
\]

(28)
6 Exercises

Exercise 6.1. Let us consider the mixed formulation (Eq. 29) of the 2D Poisson problem. Let $V_h \subset H(\text{div}, \Omega)$ and $W_h \subset L^2(\Omega)$ be finite dimensional subspaces. The Galerkin method applied to (Eq. 29) leads to the following discrete problem

Find $(u_h, \phi_h) \in V_h \times W_h$ such that for all $(\Psi, \varphi) \in V_h \times W_h$

\[
\begin{align*}
\int_{\Omega} u_h \cdot \Psi + \int_{\Omega} \phi_h \nabla \cdot \Psi &= 0, \\
\int_{\Omega} (\nabla \cdot u_h) \varphi &= -\int_{\Omega} f \varphi.
\end{align*}
\]

We assume that the computational domain $\Omega$ is the unit square.

![Figure 1: The computational domain and its triangulation.](image)

1. Let us assume that $V_h = \text{span} \{\Psi_1, \Psi_2, \ldots, \Psi_n\}$ and $W_h = \text{span} \{\varphi_1, \varphi_2, \ldots, \varphi_m\}$.

Write the global matrices and the corresponding right hand sides.

2. The latter linear system can be written as

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{u} \\
\mathbf{\phi}
\end{pmatrix} =
\begin{pmatrix}
\mathbf{f} \\
\mathbf{g}
\end{pmatrix}
\]

\[(30)\]

(a) We assume that $A$ is invertible. Show that the system 30 has a unique solution if the matrix $BA^{-1}B^T$ is invertible. (This method is called the Schur Complement)

(b) If $A$ is positive definite, show that $BA^{-1}B^T$ is also positive definite.

(c) Use the symmetry of $A$ to show that $BA^{-1}B^T$ is also a symmetric matrix.
3. We consider a triangulation $\mathcal{T}_h$ constructed as the following: we take a uniform mesh of $n \times n$ subsquares, where every subsquare is divided into two triangles as in the figure (Fig. 1).

We consider $V_h$ to be the space of piecewise linear vector fields and $W_h$ the space of continuous linear scalar fields.

(a) For $T \in \mathcal{T}_h$, a given triangle, construct a function $v \in W_h$ such that $\int_T v = 0$.

Show how to construct a piecewise linear function that is orthogonal to all piecewise constants.

(b) Show that $\int_{\Omega} v \nabla \cdot \varphi = 0$, $\forall \varphi \in V_h$.

(c) Show that the stiffness matrix is singular.

(Hint: Show that the function $(0, v)$ belongs to the kernel of the stiffness matrix).

Exercise 6.2 (The Checkerboard instability). In this exercise, we study the Stokes equation with the second weak formulation, when the inf-sup condition is not fulfilled. The computational domain will be the unit square, with a uniform mesh of $n \times n$ subsquares $Q_h = \bigcup_{i,j=1}^{n} Q_{i,j}$, as described in figure (Fig. 2). Let us recall that the variational problem can be written as

\[
\begin{cases}
\int_{\Omega} \nabla u : \nabla \Psi - \int_{\Omega} p \nabla \cdot \Psi = \int_{\Omega} f \cdot \Psi \\
\int_{\Omega} \varphi \nabla \cdot u & = 0
\end{cases}
\]

Find $(u, p) \in H^1_0(\Omega) \times L^2(\Omega)$ such that $\forall (\Psi, \varphi) \in H^1_0(\Omega) \times L^2(\Omega)$

\[
\begin{cases}
\int_{\Omega} \nabla u : \nabla \Psi - \int_{\Omega} p \nabla \cdot \Psi = \int_{\Omega} f \cdot \Psi \\
\int_{\Omega} \varphi \nabla \cdot u & = 0
\end{cases}
\]

(31)

We consider $V_h$ to be the space of continuous linear vector fields and $W_h$ the space of piecewise
constants scalar fields, i.e.

\[ V_h = \{ u_h \in [C^0(\bar{\Omega})]^2; \forall T \in \mathcal{T}_h, \ u_h \in [P_1]^2, \ u_h|_{\partial \Omega} = 0 \} \]

\[ W_h = \{ \phi_h \in L^2(\Omega) \cap C^0(\bar{\Omega}); \forall T \in \mathcal{T}_h, \ \phi_h \in P_1, \ \int_{\Omega} \phi_h = 0 \} \]

For better clarity, we define the function \( \varphi \in W_h \), which is a piecewise constant, by its value in the center of our meshes, i.e. \( \varphi := \varphi_{i+\frac{1}{2},j+\frac{1}{2}} \), while for any function \( \Psi \in V_h \) is defined by its values on the vertices of our mesh, i.e. \( \Psi_{i,j} = \left( \Psi^x_{i,j}, \Psi^y_{i,j} \right) \).

The aim of this exercise is to construct a non zero scalar-valued function \( q \) such that \( \int_{\Omega} q \nabla \cdot \Psi = 0 \), \( \forall \Psi \in V_h \).

1. Show that for all \( (\Psi, \varphi) \in V_h \times W_h \), we have

\[ \int_{\Omega} \varphi \nabla \cdot \Psi = -\frac{1}{n^2} \sum_{2 \leq i,j \leq n-1} \left( \Psi^x_{i,j} (\delta^x[\varphi])_{i,j} + \Psi^y_{i,j} (\delta^y[\varphi])_{i,j} \right) \] (32)

where the operators \( \delta^x \) and \( \delta^y \) are defined as

\[ (\delta^x[\varphi])_{i,j} := \frac{n}{2} \left( \varphi_{i+\frac{1}{2},j+\frac{1}{2}} - \varphi_{i-\frac{1}{2},j+\frac{1}{2}} + \varphi_{i+\frac{1}{2},j-\frac{1}{2}} - \varphi_{i-\frac{1}{2},j-\frac{1}{2}} \right) \]

\[ (\delta^y[\varphi])_{i,j} := \frac{n}{2} \left( \varphi_{i+\frac{1}{2},j+\frac{1}{2}} + \varphi_{i-\frac{1}{2},j+\frac{1}{2}} - \varphi_{i+\frac{1}{2},j-\frac{1}{2}} - \varphi_{i-\frac{1}{2},j-\frac{1}{2}} \right) \]

and \( \Psi = (\Psi^x, \Psi^y) \)

2. Prove that \( \int_{\Omega} \varphi \nabla \cdot \Psi = 0 \), \( \forall \Psi \in V_h \) if and only if for any \( 1 \leq i, j \leq n-1 \) we have

\[ \begin{cases} 
\varphi_{i+\frac{1}{2},j+\frac{1}{2}} = \varphi_{i-\frac{1}{2},j-\frac{1}{2}} \\
\varphi_{i-\frac{1}{2},j+\frac{1}{2}} = \varphi_{i+\frac{1}{2},j-\frac{1}{2}} 
\end{cases} \]

3. Construct such a function by using the values \(-1\) and \(+1\). Such oscillatory function is known as a spurious mode.