The Vlasov-Poisson system derived from the Vlasov-Maxwell equations is a common model for the simulation of electrostatic plasmas and charged particle dynamics. For a given electric field $E$ and a magnetic field $B$ the transport of a six dimensional phase space density $f$ is described by the Vlasov equation (1).

\[
\frac{\partial f(x,v,t)}{\partial t} + v \cdot \nabla_x f(x,v,t) - [E(x,t) + v \times B(x,t)] \cdot \nabla_v f(x,v,t) = 0.
\]  

(1)

The values of the density $f$ stay constant along the characteristic curves $t \mapsto Z(t) = (X(t),V(t))$ of eq. (1), which satisfy

\[
\frac{d}{dt} X(t) = V(t) \quad \text{and} \quad \frac{d}{dt} V(t) = -[E(x,t) + v \times B(x,t)].
\]  

(2)

By following these characteristics the initial values $f(x,v,t = 0)$ for every $(x,v) \in \mathbb{R}^6$ can be transported over time without any form of diffusion such that $f(x = X(0),v = V(0),0) = f(X(t),V(t),t)$. For a known initial value $f(\cdot,\cdot,t = 0)$ the only source of error or “numerical diffusion” is the time discretization by a time integrator. A suitable time integrator for following eq. (2) with good properties is the Boris scheme [4]. Chosing a set of initial points in phase space according to a quadrature rule and transporting by a time integrator. A suitable time integrator for following eq. (2) with good properties is the Boris scheme [4].

Monte Carlo (MC) integration provides numerical quadrature independently of the dimension. Therefore, we define a stochastic process $Z(t) := (X(t),V(t))$ with probability density $f$, which satisfies eq. (2) by construction. The MC samples are then identically independently distributed realizations of $Z(t)$, which are easily obtained at $t = 0$ since $f(\cdot,\cdot,0)$ is known. For later times these samples are obtained by solving the ODE (2), which coincides with the equations of motion of charged particles. Therefore, we call the samples particles, knowing that they neither represent physical particles, nor can any additional physical effect be applied directly on these particles. It first has to be modelled by modification of the underlying equation (1).

To simulate the particles’ self consistent field, the Vlasov equation is coupled to the Poisson equation $-\Delta \Phi = \rho$ for the electric potential $\Phi$ by the charge density $\rho_{\text{ele}}(x,t) = \int f(x,v,t)dv$ resulting in the electric field $E := -\nabla_x \Phi$. For the solution of the Poisson equation, a grid is introduced yielding the discretization $\rho_h$ of the density $\rho$. The projection of the MC particles onto the grid defines an estimator $\hat{\rho}_h$ for the charge density, which is used resulting in the classical PIC. The average error can be quantified by the mean squared error (MSE):

\[
MSE[\hat{\rho}_h(x)] = \mathbb{V}[\hat{\rho}_h(x)] + |\rho_h(x) - \rho(x)|^2.
\]  

(3)

The bias $|\rho_h(x) - \rho(x)|$ represents the field discretization error and the variance the particle noise, which decreases inversely proportional to the number of markers. For the spectral Particle in Fouriers (PIF), cf. [1, 2], and the multivariate normal $\rho(x) = (2\pi)^{-d/2}e^{-x^2/2}$ in $d = 1, 2, 3$ dimensions, defining $X \sim \mathcal{N}(0,I_d)$ we can estimate the Fourier coefficients $\rho_k$ of $\rho$ by estimating the expectation

\[
\rho_k = \int_{\mathbb{R}^d} \rho(x)e^{-ik \cdot x}dx = \mathbb{E}[e^{-ik \cdot X}] = e^{-\frac{|k|^2}{2}}.
\]  

(4)
For $m$ Fourier modes $\rho_n = S_m(\rho)$ is the $m^{th}$ order Fourier polynom of $\rho$ resulting in the bias to be the corresponding Fourier residual. The corresponding variance of the Fourier coefficients reads $\mathbb{V} [e^{-i k \cdot X}] = (1 - e^{-k^2})$. Here with higher mode number $k$ the variance increases, which corresponds to more noise in the small scales. If we do not explicitly Fourier filter some modes the number of discrete modes $k$ increases exponentially with the dimension $d$, bringing the curse of dimensionality back into the MC integration, by increasing the overall variance of $\hat{\rho}_n$ depending on all coefficients.

It remains to conclude that the error of particle grid methods like PIC or PIF is comprised by three components: the time discretization error, the field discretization error (bias) and the MC or particle noise (variance). The central limit theorem guarantees only in the many particle limit a normally distributed MC error which is a necessity for modeling and using the particle noise of poorly resolved simulations as Brownian motion. So the introduction of space charge lets the Lagrangian particles also suffer from the curse of dimensionality and an additional diffusion of unknown form in every dimension thus giving noiseless Eulerian methods an advantage in low dimensions. Nevertheless variance reduction methods, such as control variates provide a possible remedy.

References


