Exact solution of Helmholtz equation in 2D.

Lectures: Dr. O. Maj | Exercises: L. Guidi

Problem [2D Helmholtz]. We want to compute the exact solution of

\[
\begin{cases}
\varepsilon^2 \Delta u^\varepsilon (x) + u^\varepsilon (x) = 0, & x = (x_1, y) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
u^\varepsilon (0, y) = u^\varepsilon_*(y), \\
B^\varepsilon u^\varepsilon (y) = 0,
\end{cases}
\]

(1)

where the boundary operator \( B^\varepsilon \) is given by

\[
B^\varepsilon u^\varepsilon (y) = \frac{1}{2\pi \varepsilon} \int e^{\frac{i}{\varepsilon} y \cdot N_y} \left( \partial_{x_1} \hat{u}^\varepsilon (x_1, N_y) - \frac{i}{\varepsilon} (1 - N_y^2)^{1/2} \hat{u}^\varepsilon (x_1, N_y) \right) dN_y \bigg|_{x_1=0},
\]

and with boundary condition given on \( \Sigma = \{ x_1 = 0 \} \) (“antenna launcher”)

Exercise I.. We recall the expression of the semiclassical Fourier transform:

\[
\hat{\phi} (\xi) = \left( F_\varepsilon \phi \right) (\xi) = \int_{\mathbb{R}^d} e^{-\frac{i}{\varepsilon} \xi \cdot x} \phi (x) dx,
\]

Its inverse - whenever it is well defined - reads:

\[
\left( F_\varepsilon^{-1} \right) \hat{\phi} (x) = \frac{1}{(2\pi \varepsilon)^d} \int_{\mathbb{R}^d} e^{\frac{i}{\varepsilon} x \cdot \xi} (F_\varepsilon \phi) (\xi) d\xi.
\]

1. Derive and solve an equation for \( \hat{u}^\varepsilon (x, N_y) \) (namely, the semiclassical Fourier transform of \( u^\varepsilon \) with respect to the variable \( x_2 = y \), where we denote by \( N_y \) the corresponding conjugate variable).

Email: lorenzo.guidi@ipp.mpg.de

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2. Write the formula for the inverse Fourier transform with respect to $N_y$ of the solution of the first point.

**Remark.** *Keep in mind the expression in Exercise I.c of the first Exercise Sheet!*

**Exercise II.** Use a similar technique as in Exercise I in order to find a solution of the following Cauchy problem (Schrödinger equation):

\[
\begin{align*}
&i\varepsilon \frac{\partial u^\varepsilon}{\partial t}(t,x) + \varepsilon^2 \Delta u^\varepsilon(t,x) + u^\varepsilon(t,x) = 0, \\
&u^\varepsilon(0, x) = u_0^\varepsilon(x).
\end{align*}
\]  

(2)

**Remark.** *Be careful: with respect to which variable(s) do we need to apply the semi-classical Fourier transform?*