An error estimate for Hamiltonian gyrokinetics

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Kinetic Theory and Fluid Mechanics: theoretical and computational aspects

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What is gyrokinetics?

- **Full kinetic model, Vlasov equation:**
  \[ x \in \mathbb{R}^3, \ v \in \mathbb{R}^3 \]
  \[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \frac{e}{m} (E + v \times B) \cdot \frac{\partial f}{\partial v} = 0. \]

- **Gyrokinetic (GK) model:**
  \[ r \in \mathbb{R}^3, \ q_\parallel \in \mathbb{R}, \]
  \[ \frac{\partial F}{\partial t} + \frac{dr}{dt} \cdot \frac{\partial F}{\partial r} + \frac{dq_\parallel}{dt} \frac{\partial F}{\partial q_\parallel} = 0. \]

- \[ f(t, x, v) = F(t, \tau^{-1}(x, v, E, B)) + \text{error}. \]
Fusion and space physics

Tokamak and stellarator concepts.

Earth magnetic field and solar wind.
Some GK codes for HPC

ORB5 [1], GYRO [2], GENE [3], GYSELA [4], GTC [5].


Constant field case

\[ \frac{dx}{dt} = v, \quad \frac{dv}{dt} = \frac{e}{m} \left[ v \times B + E \right]. \]

Suppose \( B = e_z, E = \text{const.} \), then \( x_\parallel = z \) and \( x_\perp = (x, y) \):

\[ x_\parallel(t) = x_\parallel(0) + v_\parallel(0) \ t + \frac{e}{m} E_\parallel \frac{t^2}{2}, \]

\[ x_\perp(t) = x_\perp(0) - \left| v_\perp(0) \right| \frac{e_2}{\omega_c} + \left| v_\perp(0) \right| \left[ \frac{\sin(\omega_c t)}{\omega_c} e_1 + \frac{\cos(\omega_c t)}{\omega_c} e_2 \right] + \frac{E \times B}{B^2} t. \]

▶ In the constant-field case the gyro-center \( r \) is well-defined.

▶ Non-homogeneous case:
1) where is the slow center of motion?
2) what are the equations of motion?
GK theory

▶ Averaging on the level of Vlasov: \( f = f_0 + \varepsilon f_1 + \ldots \)


▶ Classical averaging on the level of the characteristics:

\[
\frac{dx}{dt} = v, \quad \frac{dv}{dt} = \frac{e}{m} \left[ v \times B(x, t) + E(x, t) \right], \quad \frac{d}{dt} f(x(t), v(t), t) = 0.
\]


▶ "Variational averaging" on the level of the Lagrangian:


A lot of mathematical progress


Overview

- No mathematical results on variational averaging (surprising!)
- Complicated framework: Lagrangian mechanics, exterior calculus, differential forms and Lie transforms (not easily accessible)
- No standard textbook on GK is available
- Formal results are scattered over many papers, no consistent notation.

This work:

- Existence of a gyro-transformation $\tau$, algebraic in the generators;
- Simple framework: the tangent map
- Gauge-invariance;
- Strong error estimate with respect to the exact dynamics.
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### Problem statement

#### Exact dynamics

\[
\begin{align*}
\frac{dx}{dt} &= v, & x(t_0) &= x_0, \\
\frac{dv}{dt} &= \frac{v \times B_0(\varepsilon_B x)}{\varepsilon} + v \times B_1(x, t) + E(x, t), & v(t_0) &= v_0.
\end{align*}
\]

(1)

Here, we assume \( x, x_0 \in \Omega_x \subset \mathbb{R}^3, \) \( v, v_0 \in \Omega_v \subset \mathbb{R}^3 \) with \( \Omega = \Omega_x \times \Omega_v \) open and bounded and \( 0 < \varepsilon \leq \varepsilon_{\text{max}}. \)

- Hamiltonian system with Poisson bracket (non-canonical symplectic).
- Variational formulation in terms of phase space Lagrangian
- Average the fast scale due to \( v \times B_0 \) on the level of the Lagrangian
  \( \implies \) keep the Hamiltonian structure (conservation laws!)
Variational formulation

The variational formulation of (1) is based on a Lagrangian function, defined on the tangent bundle of the underlying manifold.

The tangent bundle

- Let $M \subset \mathbb{R}^n$ open with points $m \in M$.
- Coordinate chart $\varphi : U \subset \mathbb{R}^n \rightarrow M$, $q \mapsto m$ ($M$ is thus an $n$-dimensional differentiable manifold). $q$ are called coordinates of $M$ under the chart $\varphi$.
- Tangent space $TM_m$ at point $m \in M$ is the space of all vectors originating from $m$, hence $TM_m = \mathbb{R}^n$.
- More precisely $TM_m$ contains equivalence classes of curves through $m$, two curves being equivalent when they are tangent at $m$.
- Tangent bundle:

$$TM := \bigcup_{m \in M} TM_m.$$
Coordinates on $TM$ from the chart $\varphi : U \subset \mathbb{R}^n \to M$

For $I \subset \mathbb{R}$ open, let $c : I \to U$ denote a curve in $U$ with $c(0) = q$; $\varphi(c)$ is a curve on the manifold $M$ passing through $m$ at $t = 0$.

Tangent at $q$ in the coordinate space is $\dot{q} := \frac{d}{dt} c(0)$; in $TM_m$ the tangent is

$$\xi = \frac{d\varphi(c(t))}{dt} \bigg|_{t=0} = \sum_j \frac{\partial \varphi}{\partial q_j} \bigg|_{c(0)} \frac{dc_j(0)}{dt} = D\varphi(q) \cdot \dot{q}.$$  

Any $\xi \in TM_m$ can be written as $D\varphi(q) \cdot \nu$ for some $\nu \in \mathbb{R}^n$.

$TM_m$ is the image of the Jacobian $D\varphi(q)$; columns of $D\varphi(q)$ are a basis of $TM_m$, which we denote by $\partial_j := \frac{\partial \varphi}{\partial q_j}$.

The coefficients of a tangent vector $\xi \in TM_m$ in this basis are denoted by $\dot{q}$,

$$\xi = D\varphi(q) \cdot \dot{q} = \sum_j \dot{q}_j \partial_j.$$  

Coordinates on $TM$ induced by $\varphi$ are denote by $(q, \dot{q})$.  

Dual of $\mathcal{T}M$: cotangent bundle

The cotangent bundle

- Cotangent space $T^*M_m$ contains linear forms $\gamma: TM_m \to \mathbb{R}$.
- The dual basis $d_i \in T^*M_m$ is defined by the property $d_i(\partial_j) = \delta_{ij}$.
- For Jacobians we have $D\varphi^{-1}D\varphi = I_n$, thus the lines of $D\varphi^{-1}$ are the dual basis, $d_i := \nabla \varphi_i^{-1}$.
- For general $\gamma \in T^*M_m$ and $\xi \in TM_m$,

$$\gamma(\xi) = \sum_i \gamma_i \, d_i(\xi) = \sum_{ij} \gamma_i \, \dot{q}_j \, d_i(\partial_j) = \sum_{ij} \gamma_i \, \dot{q}_j \, \delta_{ij} = \gamma \cdot \dot{q}.$$

- Cotangent bundle:

$$T^*M := \bigcup_{m \in M} T^*M_m^*.$$

Natural pairing between $TM_m$ and $T^*M_m$ is written as scalar product in the bases induced by the chart $\varphi$. 
Lagrangian and variational principle

We consider a particular class of Lagrangians $L : TM \to \mathbb{R}$ which, in local coordinates $(q, \dot{q})$ defined by some chart $\varphi : U \subset \mathbb{R}^n \to M$, can be written as

$$L(q, \dot{q}) = \gamma(q) \cdot \dot{q} - H(q).$$

Here, $H : M \to \mathbb{R}$ is called the Hamiltonian and $\gamma \in T^*M$ is the symplectic form.

Action principle

Let us denote curves in coordinate space $U$ by $q(s)$, or more precisely by $q : I \to U$ for $I \subset \mathbb{R}$ open. Define "action functional" on the space of curves,

$$A[q] := \int_I L\left(q(s), \frac{d}{ds} q(s)\right) ds.$$

Variational (action) principle $\delta A / \delta q = 0$ yields the Euler-Lagrange equations

$$\frac{\partial L}{\partial q} - \frac{d}{ds} \frac{\partial L}{\partial \dot{q}} = 0.$$
Symplectic systems

For $L(q, \dot{q}) = \gamma(q) \cdot \dot{q} - H(q)$ we obtain

$$\omega \cdot \frac{dq}{ds} = \frac{\partial H}{\partial q},$$

where $\omega := (D\gamma)^T - D\gamma$ is called the Lagrange matrix. If $J = \omega^{-1}$ exists, then

$$\frac{dq}{ds} = \{q, H\},$$

where $\{G, H\} := \frac{\partial G}{\partial q} \cdot J \cdot \frac{\partial H}{\partial q}$ denotes the Poisson bracket. The bracket is bilinear, anti-symmetric and satisfies the Jacobi identity

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0.$$

Such systems where $\omega$ are called non-canonical symplectic systems (subclass of Hamiltonian systems).

- $\frac{d}{ds} H(q(s)) = 0$.
- Conservation of phase space volume $\sqrt{\det \omega}$.
- Conservation of Casimirs and momentum maps.
- Desired: exact conservations on the discrete level. A well-known symplectic integrator is the Störmer-Verlet scheme.
The guiding-center problem

- Dynamical fields $E$ and $B_1$ in (1) are zero (autonomous system).
- Coordinate space $U = \Omega = \Omega_x \times \Omega_v$.
- $A_0$ is the vector potential s.t. $B_0 = \nabla \times A_0$.

The problem of averaging reduces to the so-called guiding-center (GC) problem,

$$L_a = \left( v + \frac{A_0(x)}{\varepsilon} \right) \cdot \dot{x} - \frac{|v|^2}{2}.$$ 

In terms of the generic Lagrangian we have

$$\gamma = \gamma_a = \left( v + \frac{A_0(x)}{\varepsilon}, 0, 0, 0 \right), \quad H = H_a = \frac{|v|^2}{2}. \quad (2)$$

The velocity components of the symplectic form are zero.


The full problem

- Dynamical fields $E$ and/or $B_1$ in (1) are NOT zero (non-autonomous system).

- "Extended" coordinate space $U = \Omega \times \mathbb{R}^2$ with coordinates $q = (x, v, t, w)$.

- Electromagnetic potentials $A_1$ and $\phi$ s.t. $B_1 = \nabla \times A_1$, $E = -\nabla \phi - \frac{\partial A_1}{\partial t}$.

The Lagrangian is of the generic form:

$$
L = \left( v + \frac{A_0(x)}{\varepsilon} + A_1(x, t) \right) \cdot \dot{x} - w \dot{t} - \frac{|v|^2}{2} - \phi(x, t) + w \cdot \gamma(q) \cdot \dot{q} - H_{\text{ext}}(q).
$$

Euler-Lagrange equations yield $\frac{d}{ds} t = 1$ and thus $t = s$.

The exact dynamics occur on the hyper-plane $H_{\text{ext}} = 0$ or $w = H := \frac{|v|^2}{2} + \phi$,

$$
L_1 := L \bigg|_{w=H} = \left( v + \frac{A_0(x)}{\varepsilon} + A_1(x, t) \right) \cdot \dot{x} - H \dot{t}.
$$

The "extended Lagrangian" (3) is the starting point for variational averaging.
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The tangent map

- Again $M \subset \mathbb{R}^n$ with chart $\varphi : U \subset \mathbb{R}^n \to M$, $q \mapsto m$.
- Different chart $\psi : V \subset \mathbb{R}^n \to M$, $l \mapsto m$.
- The map $\tau : V \to U$, $l \mapsto q$ given by $q = \tau(l) = \varphi^{-1} \circ \psi(l)$ defines a change of coordinates for the manifold $M$.
- $\tau$ is a diffeomorphism with Jacobian $D\tau = D\varphi^{-1}D\psi$.

Transformation of tangents

For $\xi \in TM_m$ we have

$$\xi = D\varphi(q) \cdot \dot{q} = D\psi(l) \cdot \dot{l}.$$ 

Therefore,

$$\dot{q} = D\varphi(\tau(l))^{-1} D\psi(l) \cdot \dot{l} = D\tau(l) \cdot \dot{l}.$$ 

We introduce the "tangent map" $T\tau : TV \to TU$:

$$(q, \dot{q}) = T\tau(l, \dot{l}) := (\tau(l), D\tau(l) \cdot \dot{l}).$$
Preservation of symplectic structure

The tangent map defined by (4) is the principal tool for variational averaging:

\[ L_1(q, \dot{q}) = \gamma_1(q) \cdot \dot{q} = \gamma_1(\tau(l)) \cdot D\tau(l) \cdot \dot{l} = \hat{\gamma}_1(l) \cdot \dot{l}. \]  

(5)

Transformation of cotangents (linear forms)

\[ \hat{\gamma}_1 = D\tau^T \cdot (\gamma_1 \circ \tau). \]  

(6)

▶ In (5) the generic form of the extended Lagrangian is preserved under the tangent map.

▶ From (6) the new Lagrange matrix \( \hat{\omega} = (D\hat{\gamma}_1)^T - D\hat{\gamma}_1 \) is invertible, and hence the symplectic structure preserved.
Preliminary transformation

Define a local, orthonormal basis \((e_1(x), e_2(x), b_0(x))\) such that \(b_0 \cdot e_1 \times e_2 = 1\).

\[
\mathbf{v}_{\parallel} := \mathbf{v} \cdot b_0(x),
\]

\[
\mathbf{v}_{\perp} := |b_0(x) \times \mathbf{v} \times b_0(x)| = |\mathbf{v} - \mathbf{v} \cdot b_0(x) b_0(x)|,
\]

\[
\theta := -\arctan2\left(\frac{\mathbf{v} \cdot e_2(x)}{\mathbf{v} \cdot e_1(x)}\right),
\]

such that

\[
\mathbf{v} = \mathbf{v}_{\parallel} b_0(x) + \mathbf{v}_{\perp} c_0(x, \theta), \quad c_0(x, \theta) := e_1(x) \cos \theta - e_2(x) \sin \theta.
\]

Extended Lagrangian after preliminary map

Introduce coordinates \(q' \in \Omega'_I\) with \(q' = (x, \mathbf{v}_{\parallel}, \mathbf{v}_{\perp}, \theta, t)\), then

\[
L'_I(q', \dot{q}') = \left[\mathbf{v}_{\parallel} b_0(x) + \mathbf{v}_{\perp} c_0(x, \theta) + \frac{A_0(x)}{\varepsilon} + A_1(x, t)\right] \cdot \dot{x} - \left[\frac{\mathbf{v}_{\parallel}^2 + \mathbf{v}_{\perp}^2}{2} + \phi(x, t)\right] \dot{t}.
\]
The gyro-transformation $\tau^\varepsilon$

**New ansatz: algebraic GY-transformation**

Let $\tau^\varepsilon : \Omega_{gy} \rightarrow \Omega'_1$, $q_{gy} \mapsto q'$, be a finite power series in $\varepsilon$:

$$q' = \tau^\varepsilon(q_{gy}) := q_{gy} + \sum_{n=1}^{N+1} \varepsilon^n G_n(q_{gy}). \quad (7)$$

- $N \geq 0$ denotes the order of the transformation
- $G_n : \Omega_{gy} \rightarrow \Omega'_1$ are the generators of this transformation.
- The GY-coordinates $q_{gy} = (q_{gy,i})_{1 \leq i \leq 7}$ and the generators $G_n = (G_n,i)_{1 \leq i \leq 7}$ are denoted by

\[
\begin{align*}
(q_{gy,i})_{1 \leq i \leq 3} &= r, & q_{gy,4} &= q_\parallel, & q_{gy,5} &= q_\perp, & q_{gy,6} &= \alpha, & q_{gy,7} &= t, \\
(G_n,i)_{1 \leq i \leq 3} &= \varrho_n, & G_n,4 &= G_n^\parallel, & G_n,5 &= G_n^\perp, & G_n,6 &= G_n^\theta, & G_n,7 &= 0.
\end{align*}
\]
Moreover, from the definition of the tangent map one obtains

\[ \dot{x} = \dot{r} + \sum_{n=1}^{N+1} \varepsilon^n \dot{q}_n(q_{gy}, \dot{q}_{gy}), \]

\[ \dot{q}_n := \frac{\partial q_n}{\partial q_{gy}} \cdot \dot{q}_{gy}. \]
Tangent map in the extended Lagrangian

\[ L^\varepsilon(q_{gy}, \dot{q}_{gy}) := L'_1(q', \dot{q}') = L'_1(T\tau^\varepsilon(q_{gy}, \dot{q}_{gy})) = L'_1(\tau^\varepsilon(q_{gy}), D\tau^\varepsilon(q_{gy}) \cdot \dot{q}_{gy}) \].

Assume regular potentials, then Taylor-expand:

\[ L^\varepsilon = \frac{L_{-1}}{\varepsilon} + L_0 + \varepsilon L_1 + \ldots + \varepsilon^N L_N + \varepsilon^{N+1} L_{N+1} + \ldots \]

\[ =: L^{(N)}_{gy} \]

Example: guiding-center problem, first order \( N = 1 \) only in \( A_0 \)

\[ L^\varepsilon = \left( v_{\|} b_0(x) + v_{\perp} c_0(x, \theta) + \frac{A_0(x)}{\varepsilon} \right) \cdot \dot{x} - \frac{v_{\|}^2 + v_{\perp}^2}{2} \]

\[ = \left( v_{\|} b_0(r + \varepsilon \varrho_1) + v_{\perp} c_0(r + \varepsilon \varrho_1, \theta) + \frac{A_0(r + \varepsilon \varrho_1)}{\varepsilon} \right) \cdot (\dot{r} + \varepsilon \dot{\varrho}_1) - \frac{v_{\|}^2 + v_{\perp}^2}{2} \]

= Taylor \ldots

Choose \( \varrho_1 \) such that \( \theta \) is eliminated from the Lagrangian: \( \frac{\partial L^\varepsilon}{\partial \theta} - \frac{d}{dt} \frac{\partial L^\varepsilon}{\partial \dot{\theta}} = 0 \).
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Definition

The guiding-center (GC) Lagrangian is defined as

\[ L_{gc}(q_{gy}, \dot{q}_{gy}) := \left[ q_{\parallel} b_0(r) + \frac{A_0(r)}{\varepsilon} \right] \cdot \dot{r} + \varepsilon \left( \frac{q_{\perp}^2}{2|B_0(r)|} \right) \frac{\dot{\alpha}}{\bar{\mu}_0} - \left[ \frac{q_{\parallel}^2}{2} + \frac{q_{\perp}^2}{2} \right] \dot{t}. \]

Definition

Two Lagrangians \( L^*, L \) defined on \( TU \) are equivalent, \( L^* \sim L \), if there exists a function \( S : U \to \mathbb{R} \) such that \( L^* = L + \partial S / \partial q \cdot \dot{q} \).

Assumption

For \( N \geq 0 \) we suppose \( A_0 \in C^{N+3}(\Omega_x), A_1 \in C^2(\mathbb{R}; C^{N+2}(\Omega_x)) \) for the vector potential and \( \phi \in C^1(\mathbb{R}; C^{N+2}(\Omega_x)) \) for the electrostatic potential.
Proposition

\( (\varepsilon_B = 1.) \) With the above regularity of \( A_0, A_1, \phi \) and with \( \nabla B_0 = O(1) \), the Lagrangian \( L^\varepsilon \) is equivalent to the following series expansion,

\[
L^\varepsilon \sim \frac{1}{\varepsilon} L_{-1} + L_0 + \sum_{n=1}^{N} \varepsilon^n L_n + O(\varepsilon^{N+1}),
\]

with

\[
L_{-1} = A_0 \cdot \dot{r},
\]

\[
L_0 = (q_\parallel b_0 + q_\perp c_0 - \varrho_1 \times B_0 + A_1) \cdot \dot{r} - \left( \frac{q_\parallel^2}{2} + \frac{q_\perp^2}{2} + \phi \right) \dot{t},
\]

\[
L_{1 \leq n \leq N} = (G_\parallel b_0 + G_\perp c_0 - \varrho_{n+1} \times B_0 - \varrho_n \times B_1) \cdot \dot{r} - (q_\parallel G_\parallel + q_\perp G_\perp - \varrho_n \cdot E) \dot{t} + Q_n(q_{gy}) \cdot \dot{r} - \mathcal{L}_n(q_{gy}, \dot{q}_{gy}),
\]

where \( Q_n : \Omega_{gy} \to T^* \Omega_{gy} \) and \( \mathcal{L}_n : T\Omega_{gy} \to \mathbb{R} \).
Proposition

(N = 1.) There exist generators $G_1 \in C^2(\Omega_{gy})$, $G_2 \in C^1(\Omega_{gy})$ such that

$$L^\varepsilon \sim L_{gc} + A_1 \cdot \dot{r} - \phi \dot{t} + O(\varepsilon^2).$$

Proposition

(N = 2.) There exist generators $G_1 \in C^3(\Omega_{gy})$, $G_2 \in C^2(\Omega_{gy})$, $G_3 \in C^1(\Omega_{gy})$ such that

$$L^\varepsilon \sim L_{gc} + A_1 \cdot \dot{r} - \phi \dot{t}$$

$$+ \varepsilon^2 \left[ - \frac{q_\perp^2}{2|B_0|^2} \left( B_1 \cdot b_0 + q_\parallel (\nabla \times b_0) \cdot b_0 \right) + \frac{q_\perp}{|B_0|} \left\langle G_1^\perp \right\rangle \right] \dot{\alpha} + O(\varepsilon^3).$$

For example:

$$x = r + \varrho_1 + \ldots$$

$$\varrho_1 = \frac{q_\perp}{|B_0|} \left( e_1 \sin \theta + e_2 \cos \theta \right).$$
Exact dynamics in the new coordinates

\[
\begin{cases}
\frac{dr}{dt} = \frac{1}{B^*_\|} \left( q\| + \varepsilon \frac{\partial H_1}{\partial q\|} \right) B^* + \varepsilon \frac{1}{B^*_\|} E^* \times b_0 + O(\varepsilon^{N+2}), & r(t_0) = r_0, \\
\frac{dq\|}{dt} = \frac{1}{B^*_\|} B^* \cdot E^* + O(\varepsilon^{N+1}), & q\| (t_0) = q\|_0 \\
\frac{d\hat{\mu}}{dt} = O(\varepsilon^{N}), & \hat{\mu}(t_0) = \hat{\mu}_0, \\
\frac{d\alpha}{dt} = \frac{|B_0|}{\varepsilon} + \frac{\partial H_1}{\partial \hat{\mu}} + O(\varepsilon^{N}), & \alpha(t_0) = \alpha_0,
\end{cases}
\]

where

\[
\begin{align*}
B^* &= \nabla \times A^*, \\
A^* &= A_0 + \varepsilon A_1 + \varepsilon q\| b_0, \\
B^*_\| &= B^* \cdot b_0, \\
E^* &= E - \hat{\mu} \nabla |B_0| - \varepsilon \nabla H_1.
\end{align*}
\]
Averaged dynamics

Truncate the error terms:

\[
\begin{align*}
\frac{d\bar{r}}{dt} &= \frac{1}{B^*_\parallel} \left(\bar{q}_\parallel + \varepsilon \frac{\partial H_1}{\partial q_\parallel} \right) B^* + \varepsilon \frac{1}{B^*_\parallel} E^* \times b_0, \\
\frac{d\bar{q}_\parallel}{dt} &= \frac{1}{B^*_\parallel} B^* \cdot E^*, \\
\frac{d\bar{\mu}}{dt} &= 0, \\
\frac{d\bar{\alpha}}{dt} &= \frac{|B_0|}{\varepsilon} + \frac{\partial H_1}{\partial \bar{\mu}},
\end{align*}
\]

\(\mathcal{P}^\varepsilon\)

\(\bar{r}(t_0) = \bar{r}_0, \quad \bar{q}_\parallel(t_0) = \bar{q}_\parallel_0, \quad \bar{\mu}(t_0) = \bar{\mu}_0, \quad \bar{\alpha}(t_0) = \bar{\alpha}_0,\)

Decoupling of the fast scale

The slow variables \(\bar{z}_{sl} := (\bar{r}, \bar{q}_\parallel, \bar{\mu})\) are decoupled from the solution for the gyro-angle \(\bar{\alpha}(t)\).
Slow variables

Exact and averaged subproblems for the slow variables:

\[
\begin{align*}
(P_{\varepsilon}^{sl}) \quad \frac{dz_{sl}}{dt} & = U_{sl}(z_{sl}, t, \varepsilon) + \varepsilon^N S_{sl}(z_{sl}, \alpha, t, \varepsilon), \\
\quad z_{sl}(t_0) & = z_{sl,0},
\end{align*}
\]

\[
(\overline{P}_{\varepsilon}^{sl}) \quad \frac{d\overline{z}_{sl}}{dt} = U_{sl}(\overline{z}_{sl}, t, \varepsilon), \\
\quad \overline{z}_{sl}(t_0) = \overline{z}_{sl,0}.
\]

Remark

The direction field \( U_{sl} \) is independent of \( \alpha(t) \). It is a \( C^1 \)-function of \( (z_{sl}, t, \varepsilon) \). In particular, \( U_{sl} \) is Lipschitz in \( z_{sl} \), uniformly in \((t, \varepsilon)\),

\[
\| U_{sl}(y_{sl}, t, \varepsilon) - U_{sl}(z_{sl}, t, \varepsilon) \| \leq \ell_U \| y_{sl} - z_{sl} \|,
\]

where the Lipschitz constant \( \ell_U \) is independent of \((t, \varepsilon)\).
Apply Gronwall

Lemma

Consider the problems \((P_\varepsilon^{\text{sl}})\) and \((\overline{P}_\varepsilon^{\text{sl}})\) for the slow GY-variables on the interval \(t \in I = [t_0, t_1]\) with \(t_1 \leq T\), then

\[
\|\overline{z}_{\text{sl}}(t) - z_{\text{sl}}(t)\| \leq \varepsilon^N \left\| S_{\text{sl}} \right\|_{\infty, \varepsilon} \left( e^{\ell_U (t-t_0)} - 1 \right) + \|\overline{z}_{\text{sl},0} - z_{\text{sl},0}\| e^{\ell_U (t-t_0)},
\]

where \(\|S_{\text{sl}}\|_{\infty, \varepsilon} = \max_{\Omega_{\text{gy}} \times (0, \varepsilon_{\text{max}})}\) and \(\|S_{\text{sl}}\| = O(1)\) w.r.t. \(\varepsilon\).
**Main result**

**Theorem**

Let \( \overline{F} : \Omega_{gy} \rightarrow \mathbb{R}_{+} \) be the unique function which is constant along the solutions of the averaged dynamics \((\overline{P}^{\varepsilon})\), defined on the interval \( I = [t_0, t_1] \), satisfying the condition \( \overline{F}(z, t_0) = \overline{F}_0(z) \) for some given \( \overline{F}_0 \). Further, let \( f : \Omega_I \rightarrow \mathbb{R}_{+} \) stand for the unique solution of the Vlasov equation satisfying \( f(x, v, t_0) = f_0(x, v) \), with given fields \( B_0(x) \), \( B_1(x, t) \) and \( E(x, t) \).

If \( \overline{F}_0 = f_0 \circ \tau' \circ \tau^{\varepsilon} \), if moreover \( \overline{F}_0 \) is Lipschitz with constant \( \ell_0 \) and \( \overline{F}_0 \) is independent of the gyro-angle \( \alpha \), then

\[
\| \overline{F} - f \circ \tau' \circ \tau^{\varepsilon} \|_{\infty} \leq \varepsilon^N C(t) \quad \text{for } t \in I,
\]

where \( \| \cdot \|_{\infty} = \max_{\Omega_{gy}} | \cdot | \) and

\[
C(t) = \ell_0 \frac{\| S_{sl} \|_{\infty, \varepsilon}}{\ell_U} (e^{\ell_U (t-t_0)} - 1).
\]
Outline

1 What is gyrokinetics?

2 Variational averaging
   - Scaled problem
   - Lagrangian functions and action principle
   - Guiding-center problem and full problem

3 Change of coordinates
   - The tangent map
   - Preliminary transformation: the gyro-angle
   - Algebraic gyro-transformation

4 Main results
   - Lagrangians and generators
   - Strong error bound for gyrokinetics

5 Conclusion
Concluding remarks

”Gyrokinetics from algebraic transformations: existence and error bounds” will soon be available on Arxiv.

Future: look at finite-Larmor-radius effects.

Numerical verification with single particle simulations.

More numerics, numerics, numerics ...

THANK YOU!